Two Dimensional Quantum Gravity and Matrix Integrals

by

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Abstract

This survey reviews recent developments in two-dimensional holography with a focus on the proposed dualities between random matrix models and two-dimensional dilaton gravity theories. The models considered are Jackiw-Teitelboim and $\mathcal{N} = 1$ supersymmetric Cangemi-Jackiw gravities, which are respectively locally hyperbolic and flat. Euclidean partition functions admit an expansion in the Euler characteristic of the spacetime manifolds, which are matched order by order with the 't Hooft expansion of the matrix integrals. For both theories, an essential step in the computation of the bulk partition function is a reformulation in terms of topological gauge theories, yielding one-loop exact results through the Duistermaan-Heckman theorem. Matrix duals provide UV completions of the gravitational theories, and observables non-perturbative in the genus expansion parameter can be extracted through the method of orthogonal polynomials and Fredholm determinants. We conclude with a discussion of open questions, and propose several future research directions.

Contents

1 Introduction

Holography has been at the forefront of recent developments in many areas of theoretical physics with applications ranging from condensed matter theory [1, 2] to the black hole information problem [3, 4, 5, 6]. Its general form states that any CFT on $\mathbb{R} \times S^{d-1}$ is equivalent to quantum gravity on asymptotically $\mathrm{AdS}_{d+1} \times \mathcal{M}$ spacetime, where $\mathcal{M}$ is a compact manifold [7]. The special case of two bulk dimensions takes a qualitatively different form. On the gravitational side one works with asymptotically AdS geometries with an IR cutoff, the Euclidean dynamics of which is universally given by Jackiw-Teitelboim (JT) gravity [8]. The gravitational path integral is organised in a sum over hyperbolic surfaces of increasing genus, referred to as the genus expansion.

In a landmark result [9], it was shown that the JT path integral is equivalent to a matrix integral of the form:

$$\mathcal{Z} = \int dM e^{-N \operatorname{Tr} V(M)}. \tag{1}$$

where M are random $N \times N$ Hermitian matrices, interpreted as the Hamiltonian of the boundary theory dual to JT gravity. $V(M)$ is the matrix potential, which is fine tuned to match with the JT path integral. In an appropriate large N limit called the double scaling limit, the JT partition function coincides with its random matrix counterpart to all orders in the genus expansion. The same procedure has since been applied to find matrix duals to other two-dimensional theories [10, 11, 12, 13].

In section 2, we review the Euclidean gravitational path integral computation of [9], focusing on the trousers decomposition of hyperbolic surfaces and the computation of the measure over boundary fluctuations using a topological gauge theory formalism.

Section 3 introduces matrix integrals, showing how to recover the many-boundary JT partition function as connected correlators of the matrix model. The rest of the section focuses on extracting non-perturbative information using the method of orthogonal polynomials and Fredholm determinants.

Section 4 explores a different two-dimensional gravity theory, called the (supersymmetric) Cangemi-Jackiw model in [14], which is locally flat and has $\mathcal{N} = 1$ supersymmetry. Its prominent feature is a particularly simple matrix dual which is analytically exactly solvable.

To conclude, section 5 discusses open questions and future directions in the two-dimensional gravity literature.

2 JT partition function

JT gravity is defined by the Euclidean action [15, 16]

$$I = -\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}R + \int_{\partial\mathcal{M}}\sqrt{h}K\right] - \frac{1}{2}\int_{\mathcal{M}}\sqrt{g}\phi(R+2) - \int_{\partial\mathcal{M}}\sqrt{h}\phi(K-1), \tag{2}$$

where S_0 is a constant, ϕ is the dilaton, K is the extrinsic curvature and $\mathcal{M}$ is the spacetime manifold. The first two terms are the Einstein-Hilbert and the Gibbons-Hawking-York terms which are topological in $d = 2$ and are equal to the Euler characteristic $2\pi\chi$ of $\mathcal{M}$. The next two terms are non-trivial due to the dilaton.

$\mathcal{M}$ can have an arbitrary number of boundaries, along which the induced metric is fixed such that each boundary has length β_i/ϵ. The dilaton is fixed to $\phi = \gamma/\epsilon$ along $\partial\mathcal{M}$, where a choice of γ is equivalent to a choice of units, and $\epsilon \to 0$ is the holographic renormalization parameter.

2.1 Genus expansion

Following [9], decomposing the action as $I = -S_0\chi(\mathcal{M}) + I_{\mathrm{JT}}$, the n-boundary Euclidean partition function is written formally as

$$Z(\beta_1, \ldots, \beta_n) = \int [dX] e^{-I_{\mathrm{JT}}[X] + S_0\chi(\mathcal{M})}, \tag{3}$$

where the integral is over all fields X as well as over all manifolds $\mathcal{M}$ with n boundaries[1] Schematically, enumerating two-dimensional manifolds by their genus, the path integral can be written as

$$Z(\beta_1, \ldots, \beta_n) = \sum_{g=0}^{\infty}(e^{-S_0})^{2g+n-2} Z_{g,n}(\beta_1, \ldots, \beta_n) + \text{non-perturbative in } e^{-S_0}, \tag{4}$$

where $Z_{g,n}$ are the restrictions of equation (3) to manifolds with genus g. Non-perturbative contributions, such as terms of order $e^{e^{S_0}}$ are in general present. Such terms don't have an immediate geometric interpretation, it is possible that they are contributions from non-geometric phases of the UV complete theory.

The dilaton appears as a Lagrange multiplier in the action. Integrating ϕ along an imaginary contour fixes $R = -2$ and the genus expansion localizes on hyperbolic surfaces. Computing the perturbative component of the partition function therefore reduces to evaluating

$$Z_{g,n}(\beta_1, \ldots, \beta_n) = \int [dg] \exp\left(\int_{\partial\mathcal{M}}\sqrt{h}\phi(K-1)\right). \tag{5}$$

This is an integral over boundary fluctuations of the geometry and over the bulk moduli space, counting the volume of distinct Riemann surfaces at given (g, n). The factorisation is made explicit by introducing a genus-zero surface with an asymptotic boundary of renormalized length β and a geodesic boundary of length b, called the trumpet geometry. Any (g, n) hyperbolic surface can be constructed by gluing together a genus g hyperbolic surface with n geodesic boundaries and n

[1] It should be noted that it is not at all obvious that "a sum over all manifolds" is well defined. Within the context of holography where spacetime itself is emergent, one would generally expect non-geometric contributions to the path integral. Furthermore, in higher dimensions there exists no finite enumeration of surfaces according to their genus.

trumpet geometries attached to each geodesic boundary.

Denote by $V_{g,n}(b_1,\ldots,b_n)$ the volume of the moduli space of genus g hyperbolic Riemann surfaces with geodesic boundaries of lengths $\{b_1,\ldots,b_n\}$. The path integral decomposes into

$$Z_{g,n}(\beta_1,\ldots,\beta_n) = \int d\mu[b] V_{g,n}(b_1,\ldots,b_n) \prod_{i=1}^{n} Z_{\text{trumpet}}(\beta_i, b_i), \tag{6}$$

where $d\mu[b]$ accounts for twists and geodesic lengths associated with each gluing. This decomposition holds for all (g,n) except for the two special cases of $(0,1)$ and $(0,2)$, which correspond to the hyperbolic disk and the gluing of two trumpets:

$$Z_{0,1}(\beta) = Z_{\text{disk}}(\beta), \tag{7}$$

$$Z_{0,2}(\beta_1,\beta_2) = \int d\mu[b] Z_{\text{trumpet}}(\beta_1, b) Z_{\text{trumpet}}(\beta_2, b). \tag{8}$$

The problem reduces to evaluating the disk and trumpet partition functions and the volumes $V_{g,n}$.

The disk and trumpet geometries are given by the metrics

$$g = \begin{cases} d\rho \otimes d\rho + \sinh^2(\rho)\, d\theta \otimes d\theta & \text{disk}, \\ d\rho \otimes d\rho + \cosh^2(\rho) d\tau \otimes d\tau & \text{trumpet}, \end{cases} \tag{9}$$

where $\rho \in \mathbb{R}_+$ is a radial coordinate, and $\theta \sim \theta + 2\pi$ and $\tau \sim \tau + b$. The parameter b labels the length of the geodesic boundary of the cylinder. Boundary actions depend only on θ and τ, and the resulting partition functions read

$$Z_{\text{disk}}(\beta) = \int \frac{d\mu[\theta]}{SL(2,\mathbb{R})} \exp\left\{-\frac{\gamma}{2}\int_0^\beta du \left[\left(\frac{\theta''}{\theta'}\right)^2 - (\theta')^2\right]\right\}, \tag{10}$$

$$Z_{\text{trumpet}}(\beta, b) = \int \frac{d\mu[\tau]}{U(1)} \exp\left\{-\frac{\gamma}{2}\int_0^\beta du \left[\left(\frac{\tau''}{\tau'}\right)^2 + (\tau')^2\right]\right\}. \tag{11}$$

Physically, these are integrals over the boundary fluctuations of the geometry, parametrised by $\theta(u)$ and $\tau(u)$. Quotients over $SL(2,\mathbb{R})$ and $U(1)$ correspond to the isometry groups of the geometries. Measures $\mu[\theta]$ and $\mu[\tau]$ are induced from the symplectic form in the gauge theory (BF) formulation.

2.2 BF formulation

JT gravity admits a reformulation in terms of a topological gauge theory. This is a key step in the evaluation of the path integral, yielding the measures over boundary fluctuations and explicitly showing one-loop exactness.

First, one writes down the first-order bulk action

$$I[\phi, \lambda^a, e^a, \omega] = -\int_{\mathcal{M}} \left[\phi(d\omega + e^1 \wedge e^2) + \lambda^1(de^1 + \omega \wedge e^2) + \lambda^2(de^2 - \omega \wedge e^1)\right], \tag{12}$$

where e^a are the zweibeins, ω is the spin connection and the Lagrange multipliers λ^a impose the no-torsion condition. Next, one identifies the group G associated with the isometries of the on-shell geometry. For JT gravity, $G = SL(2,\mathbb{R})$. Let A be the gauge connection and B a $\mathfrak{g}$-valued scalar field. The action of $g \in G$ on the fields is

$$B \to g^{-1} B g, \quad A \to g^{-1}(d + A) g. \tag{13}$$

The curvature is $F = dA + A \wedge A$ and it transforms in the adjoint representation. Finally, let $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a bilinear form which is determined through the quadratic Casimir and generators J_A:

$$\langle J_A, J_B \rangle = h_{AB} \quad \Rightarrow \quad C_2 = h^{AB} J_A J_B. \tag{14}$$

With this setup, one proposes that the action can be written as

$$I_{\rm BF}[B, A] = -i \int_{\mathcal{M}} \langle B, F \rangle + \frac{i}{2} \int_{\partial\mathcal{M}} \langle B, A \rangle \,, \tag{15}$$

where the boundary term is required to make the variational problem well-posed in the presence of a boundary.

$sl(2,\mathbb{R})$ has three generators X, Y, Z, with Lie algebra

$$[X, Y] = 2Y, \quad [X, Z] = -2Z, \quad [Y, Z] = X, \tag{16}$$

and the following non-zero basis elements of the bilinear form

$$\langle X, X \rangle = 2, \quad \langle Y, Z \rangle = \langle Z, Y \rangle = 1. \tag{17}$$

To match with the bulk action, one makes the identifications:

$$iB = -\lambda^1 X + (\lambda^2 + \phi) Y + (\phi^2 - \phi) Z, \tag{18}$$

$$A = \frac{1}{2}\big[-e^1 X + (e^2 - \omega) Y + (e^2 + \omega) Z\big]. \tag{19}$$

This completes the BF formulation of JT gravity. As a side note, the extension to supersymmetric JT gravity is very natural, one simply adds appropriate fermionic generators to the Lie algebra.

The path integral is gauge-fixed by the Fadeev-Popov-BRST procedure [17], details of which are given in appendix 1. The key result is that the integral reduces to an integral over flat connections with measure induced by the symplectic form

$$\Omega(\eta, \omega) = \alpha \int_{\mathcal{M}} \langle \eta \wedge \omega \rangle \equiv \alpha \int_{\mathcal{M}} \langle \eta_a, \omega_b \rangle \, dx^a \wedge dx^b, \tag{20}$$

and an overall correction factor by the volume of the centre subgroup. Note that this analysis relies on an orientable $\mathcal{M}$, and a restriction on the matter content of the bulk theory.

Consider variations δA which leave the connection flat,

$$0 = \delta F = d\delta A + A \wedge \delta A + \delta A \wedge A = D\delta A. \tag{21}$$

The gauge covariant derivative about a flat connection is nilpotent, hence $\delta A = D\Theta$ for some zero-form Θ. Such variations have the same structure as gauge transformations, however since we have already gauge fixed, the only remaining variations are large gauge transformations that don't die off at $\partial\mathcal{M}$. The symplectic form reads

$$\Omega(\delta_1 A, \delta_2 A) = \alpha \int_{\mathcal{M}} \langle d\Theta_1 + [A, \Theta_1], d\Theta_2 + [A, \Theta_2]\rangle = \alpha \int_{\partial\mathcal{M}} \langle \Theta_1, d\Theta_2 + [A, \Theta_2]\rangle \,, \tag{22}$$

where the last equality follows from integrating by parts, using the Bianchi identity for the gauge algebra and setting $dA = -A \wedge A$.

As it turns out [9], the set of large gauge transformations that preserve the boundary conditions at $\partial\mathcal{M}$ are parametrised by a single function $\varepsilon(u)$. Physically, the mode $\varepsilon(u)$ generates reparametrisations $u \to u + \varepsilon(u)$ of the boundary. From a gravitational perspective, it is the mode associated with boundary fluctuations. The resulting symplectic form is

$$\Omega = \frac{\alpha}{4} \int_0^\beta du[d\varepsilon' \wedge d\varepsilon'' - 2f(u) d\varepsilon \wedge d\varepsilon'], \tag{23}$$

where $f(u)$ is related to the boundary action through $I_\partial = \gamma \int du f(u)$, and $d\varepsilon(u)$ are an infinite basis on the cotangent space about a point in the space of flat connections. As $\partial\mathcal{M} \cong S^1$ is compact, this basis has a countably infinite number of elements under Fourier decomposition,

$$d\varepsilon(u) = \sum_{n=-\infty}^{\infty} e^{-2\pi i n u/\beta} \big(d\varepsilon_n^R + i d\varepsilon_n^I\big), \tag{24}$$

with reality conditions $d\varepsilon_n^R = d\varepsilon_{-n}^R$ and $d\varepsilon_n^I = -d\varepsilon_{-n}^I$

2.3 Disk and trumpet partition functions

The Duistermaat-Heckman theorem states that a path integral over a symplectic manifold is one-loop exact if the action generates a $U(1)$ symmetry of the manifold [18]. This holds for JT gravity [19].

Evaluation of the one-loop disk and cylinder path integrals proceed by evaluating the induced measures $\mu[\theta]$ and $\mu[\tau]$. Recall that the mode $\varepsilon(u)$ generates boundary wiggles through the reparametrisation $u \to u + \varepsilon(u)$. For the disk, this corresponds to the following fluctuation about the saddle point configuration:

$$\theta(u) = \frac{2\pi}{\beta}(u + \varepsilon(u)). \tag{25}$$

The disk action to quadratic order in ε reads

$$\begin{aligned} I_\partial^{\text{disk}}[\varepsilon] &= -\frac{2\pi^2\gamma}{\beta} + \frac{2\pi^2\gamma}{\beta^2} \int_0^\beta du \left[\left(\frac{\beta}{2\pi}\right)^2 (\varepsilon'')^2 - (\varepsilon')^2 \right] \\ &= -\frac{2\pi^2\gamma}{\beta} + \gamma \frac{(2\pi)^4}{\beta^3} \sum_{n>1} (n^4 - n^2) \big[(\varepsilon_n^R)^2 + (\varepsilon_n^I)^2\big]. \end{aligned} \tag{26}$$

The three zero modes $(\varepsilon_{-1}, \varepsilon_0, \varepsilon_1)$ correspond to the $SL(2,\mathbb{R})$ isometry. In the path integral, we quotient over these modes by excluding the corresponding $d\varepsilon_i$ from the symplectic form Ω. The resulting measure, when evaluated on the saddle point configuration $f(u) = 2\pi^2/\beta^2$, is

$$\frac{d\mu[\theta]}{SL(2,\mathbb{R})} = \lim_{N\to\infty} \frac{1}{N!}(\Omega[d\varepsilon_{n>2}])^N = \prod_{n>1} \alpha \frac{8\pi^3}{\beta^2}(n^3 - n) d\varepsilon_n^R d\varepsilon_n^I. \tag{27}$$

Hence, the disk partition function reduces to an infinite product of Gaussian integrals

$$\begin{aligned} Z_{\text{disk}}(\beta) &= \exp\left(\frac{2\pi^2\gamma}{\beta}\right) \prod_{n>1} \alpha \frac{8\pi^3}{\beta^2}(n^3 - n) \int_{-\infty}^{\infty} d\varepsilon_n^R d\varepsilon_n^I \exp\left(-\gamma \frac{(2\pi)^4}{\beta^3}(n^4 - n^2)\left[(\varepsilon_n^R)^2 + (\varepsilon_n^I)^2\right]\right) \\ &= \exp\left(\frac{2\pi^2\gamma}{\beta}\right) \frac{2\gamma}{\alpha\beta} \prod_{n>0} \frac{\alpha\beta}{2\gamma n}. \end{aligned} \tag{28}$$

To regularize, consider the related sum of logs

$$\sum_{n=1}^{\infty} \left[\log\left(\frac{\alpha\beta}{2\gamma}\right) - \log(n)\right] = \log\left(\frac{\alpha\beta}{2\gamma}\right)\zeta(0) + \partial_s\, \zeta(s)\Big|_{s=0} \sim \log\left(\sqrt{\frac{\gamma}{\alpha\beta\pi}}\right), \tag{29}$$

where the last step uses Zeta function regularization $\zeta(0) = -1/2$ and $\zeta'(0) = \log\big(1/\sqrt{2\pi}\big)$. With this regularization scheme, the partition function reads

$$Z_{\text{disk}}(\beta) = \frac{1}{\sqrt{2\pi}} \left(\frac{2\gamma}{\alpha\beta}\right)^{3/2} \exp\left(\frac{2\pi^2\gamma}{\beta}\right) \tag{30}$$

The trumpet case proceeds similarly, the action to quadratic order in fluctuations is

$$I_\partial^{\text{trumpet}}[\varepsilon] = \frac{\gamma b^2}{2\beta} + \frac{(2\pi)^4}{\beta^3} \sum_{n>0} \left(n^4 + \frac{b^2}{(2\pi)^2}\right)\left[(\varepsilon_n^R)^2 + (\varepsilon_n^I)^2\right]. \tag{31}$$

The zero mode ε_0 is identified with the $U(1)$ isometry of the trumpet. The measure with saddle point $f(u) = -b^2/2\beta^2$ is

$$\frac{d\mu[\tau]}{U(1)} = \prod_{n>0} \alpha \frac{8\pi^3}{\beta^2}\left(n^3 + \frac{b^2}{(2\pi)^2} n\right) d\varepsilon_n^R d\varepsilon_n^I, \tag{32}$$

and the path integral after regularization reads

$$Z_{\text{trumpet}}(\beta, b) = \sqrt{\frac{\gamma}{\pi\alpha\beta}} \exp\left(-\frac{\gamma b^2}{2\beta}\right). \tag{33}$$

2.4 Volumes of bulk moduli

This section summarises the results of Mirzakhani [20] on the evaluation of the volumes $V_{g,n}(b)$. The central result is a recursion relation relating volumes of different (g, n).

The moduli space is defined through the Teichmüller space $\mathcal{T}(S)$, which is the set of pairs of hyperbolic surfaces and diffeomorphisms (X, f) such that $f : S \to X$. In the case ∂S is non-empty, the subspace $\mathcal{T}(S, b)$ is defined by fixing the lengths ℓ_i of boundary components $i \in \partial S$:

$$\mathcal{T}(S, b) = \{(X, f) \in \mathcal{T}(S) \mid \ell_i(X) = b_i \;\forall\, i \in \partial S\}. \tag{34}$$

Let $S_{g,n}$ to be an oriented connected surface of genus g and n boundary components. The moduli space $\mathcal{N}_{g,n}(b)$ of Riemann surfaces homeomorphic to $S_{g,n}$ is the quotient space

$$\mathcal{N}_{g,n}(b) = \mathcal{T}(S_{g,n}, b)/\mathrm{Mod}_{g,n}, \tag{35}$$

where $\mathrm{Mod}_{g,n}$ is the mapping class group of $S_{g,n}$. The essential idea is that two elements of the Teichmüller space are distinct Riemann surfaces if and only if they are not related to each other through diffeomorphisms.

Next, one introduces coordinates on $\mathcal{T}(S_{g,n}, b)$ through the trousers decomposition. The idea is to divide the surface into genus zero geometries with three boundaries (trousers), by cutting along closed curves. The number of such closed curves needed is $k = 3g-3+n$, see appendix 2 for a proof. This provides a set of $2k$ coordinates $\{\ell_1, \dots, \ell_k, \tau_1, \dots, \tau_k\}$, consisting of the lengths ℓ_i of the closed geodesics and the twists τ_i. These are called the Fenchel-Nielsen coordinates. The dimension of the associated Teichmüller space is therefore $2k$.

There is a natural symplectic form ω on $\mathcal{T}_{g,n}(b)$ which is invariant under the mapping class group. It is called the Weil-Petersson form, and in Fenchel-Nielsen coordinates takes the form

$$\omega = \sum_{i=1}^{k} d\ell_i \wedge d\tau_i. \tag{36}$$

Before proceeding with the computation of $V_{g,n}$, let us recall the expression for $Z_{g,n}$ given in equation (6). The induced measure $\mu[b]$ associated with the gluing of n trumpets is simply

$$\mu[b] = db_1 \dots db_n d\tau_1 \dots d\tau_n, \tag{37}$$

where the local coordinates take values $b_i \in \mathbb{R}_+$ and $\tau_i \in [0, b)$. Performing the integrals over the twist parameters, one obtains

$$Z_{g,n}(\beta_1, \dots, \beta_n) = \int_0^\infty b_1 \dots b_n db_1 \dots db_n V_{g,n}(b_1, \dots, b_n) \prod_{i=1}^{n} Z_{\mathrm{trumpet}}(\beta_i, b_i). \tag{38}$$

Similarly, the volume $V_{g,n}(b)$ of the moduli space is computed as

$$V_{g,n} = \int_{\mathcal{N}_{g,n}} d\mu[\omega] = \frac{1}{k!} \int_{\mathcal{N}_{g,n}} \omega^k, \tag{39}$$

however the computation is much more difficult due to the restriction to a fundamental domain. The main idea of [20] is to use a "generalized McShane identity" to circumvent having to compute the fundamental domains for each (g, n). To illustrate how this works, let us consider the one-boundary torus $\mathcal{N}_{1,1}$. This surface can be constructed by gluing a single pair of trousers to itself. The space of all such gluings is

$$\mathcal{N}^*_{1,1} = \{(X, \gamma) \mid X \in \mathcal{N}_{1,1},\ \gamma \in \{\text{closed curve on } X\}\}. \tag{40}$$

This is related to $\mathcal{T}_{1,1}$ through a quotient by the stabilizer of γ, where the particular choice of γ does not matter. In Fenchel-Nielsen coordinates, one has

$$\mathcal{N}_{1,1}^* \cong \{(\ell, \tau) \mid \ell \in \mathbb{R}_+,\ \tau \in [0, \ell)\}. \tag{41}$$

The generalized McShane identity for this case reads

$$\frac{2}{b} \sum_{\gamma} \log\left(\frac{e^{b/2} + e^{\ell}}{e^{-b/2} + e^{\ell}} \right) = 1, \tag{42}$$

where b is the length of the geodesic boundary component of the torus and ℓ is the length of the closed curve γ. Integrating this over $\mathcal{N}_{1,1}$, which can be written in terms of an integral over $\mathcal{N}_{1,1}^*$ through a projection map, yields

$$V_{1,1} = \frac{2}{b} \int_0^{\infty} d\ell \int_0^{\ell} d\tau \log\left(\frac{e^{b/2} + e^{\ell}}{e^{-b/2} + e^{\ell}} \right) = \frac{1}{24}(b^2 + 4\pi^2). \tag{43}$$

Note that the sum over γ is contained in the integral over $\mathcal{N}_{1,1}^*$. Effectively, this procedure shifts the problem of finding a fundamental domain over $\mathcal{N}_{1,1}$ to finding the Jacobian associated with the trousers decomposition. For higher order (g, n), the generalized McShane identity looks much more complicated and the computations become intractable. There is a recursion relation that allows for the computation of all higher order $V_{g,n}$ from the inputs $V_{1,1}$ and $V_{0,3} = 1$.

3 Matrix integrals

Random matrix models are defined through their partition functions

$$\mathcal{Z} = \int dM e^{-N \operatorname{Tr} V(M)}. \tag{44}$$

A particular model is specified by the matrix potential $V(M)$ and the symmetry group G under which the matrices transform. There are ten common choices of G, three Dyson [21] and seven Altland-Zirnbauer ensembles [22].

We focus on the $\beta = 2$ Dyson ensemble, which has the symmetry group $U(N)$ with M Hermitian. It turns out that this model can reproduce both JT gravity [9] and the supersymmetric Cangemi-Jackiw gravity [14] under different potentials and dictionaries. The trace of any polynomial function $V(M)$ is invariant under $U(N)$, and an invariant measure is

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{i<j}^{N} d\,\mathrm{Re}(M_{ij}) d\,\mathrm{Im}(M_{ij}), \tag{45}$$

where one simply integrates over all independent components of M. Expectation values of observables $\mathcal{O}_i(M)$ are given by

$$\langle \mathcal{O}_1(M) \dots \mathcal{O}_n(M) \rangle = \frac{1}{\mathcal{Z}} \int dM e^{-N \operatorname{Tr} V(M)} \mathcal{O}_1(M) \dots \mathcal{O}_n(M). \tag{46}$$

An observable of particular importance is $Z(\beta) = \operatorname{Tr} e^{-\beta M}$, which is related through a Laplace transform to the resolvent

$$R(E) \equiv \operatorname{Tr} \frac{1}{E - M} = -\int_0^\infty d\beta \; e^{\beta E} Z(\beta). \tag{47}$$

3.1 Genus expansion

Matrix integrals are zero-dimensional Euclidean QFTs. Expectation values can be computed in perturbation theory with Feynman diagrammes. Noting that the parameter $1/N$ is analogous to Planck's constant, one expects the theory to reorganize itself in a "classical" limit, where $N \to \infty$. This was indeed shown to be the case by 't Hooft [23].

't Hooft's argument adapted for the matrix potential

$$V(M) = \frac{1}{2} M^2 + \sum_q g_q M^q \tag{48}$$

is summarized as follows. Consider a vacuum diagramme with P propagators, V vertices and I closed loops. The number of p-point vertices V_p is related to P through

$$2P = \sum_p p V_p. \tag{49}$$

Viewing each diagramme as a tessellation of a two-dimensional surface[2], each closed loop is identified

[2]This is possible because we are restricting to vacuum diagrammes.

by a face of the polyhedron. Hence, the Euler relation applies

$$V - P + I = 2 - 2G, \tag{50}$$

where G is the genus of the surface. Each diagramme therefore scales with N as

$$N^{I+P} \prod_p (g_p N^{1-p})^{V_p} = N^{2-2G} \prod_p g_p, \tag{51}$$

where the additional factor of N^P accounts for the factor of N in front of the matrix potential and can be traced back to the rescaling of the matrices $M \to M/\sqrt{N}$. In canonical scaling, the 't Hooft limit has to be accompanied with an appropriate scaling of the couplings g_p to render finite observables. In this choice of non-canonical normalisation, no such rescaling of g_p are needed.

The matrix integral admits a perturbative large N expansion

$$\mathcal{Z} = \sum_{G=0}^{\infty} \frac{\mathcal{Z}_G}{N^{2G-2}} + \text{non-perturbative}. \tag{52}$$

The above analysis of vacuum diagrammes is easily modified for correlators of n observables by adding a boundary for each observable in the Euler relation, so that a genus G diagramme with n boundary scales as

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_{G,n} \sim N^{2-2G-n} \prod_p g_p. \tag{53}$$

Hence, observables also admit large N expansions. A quantity of interest for us is the connected correlators of resolvents:

$$\langle R(E_1) \dots R(E_n) \rangle_c = \sum_{G=0}^{\infty} \frac{R_{G,n}(E_1, \dots, E_n)}{N^{2G+n-2}} + \text{non-perturbative}. \tag{54}$$

Computation of resolvents follow from the so-called loop equations. In a basis where the matrices M are diagonal, the matrix integral up to a constant reads

$$\mathcal{Z} = \int d^N\lambda \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-N \sum_i V(\lambda_i)}. \tag{55}$$

Loop equations follow from the invariance of this integral under variations of λ,

$$0 = \int d^N\lambda \frac{\partial}{\partial \lambda_a} \left[\frac{1}{E - \lambda_a} R(E_1) \dots R(E_k) \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-N \sum_i V(\lambda_i)} \right], \tag{56}$$

for a choice of $k \geq 0$. It can be shown [24] that the loop equations reduce to

$$R_{k+2}(E, E, I) + \sum_{i=1}^{k} \frac{\partial}{\partial E_i} \frac{R_k(E, I \setminus \{E_i\}) - R_k(I)}{E - E_i} = N(V'(E) R_{k+1}(E, I) - P_k(E; I)), \tag{57}$$

where $I = \{E_1, \ldots, E_k\}$, and $R_k(I) = \langle R(E_1) \ldots R(E_k) \rangle_c$, and

$$P_k(E; I) = \left\langle \operatorname{Tr} \frac{V'(E) - V'(M)}{E - M} \prod_{i=1}^{k} R(E_i) \right\rangle_c . \tag{58}$$

In the large N limit, the set of loop equations can be solved perturbatively by noting the genus expansions given by equation (54) and

$$P_k = \sum_{G=0}^{\infty} N^{1-2G-n} P_{G,k}, \tag{59}$$

where the additional factor of N^{-1} in P_k is due to it having $k+1$ insertions.

3.2 Spectral density

The spectral density $\rho(\lambda) \equiv \operatorname{Tr} \delta(\lambda I - M)$ is related to the resolvent through

$$\operatorname{Tr} e^{-\beta M} = \int_\Omega d\lambda \rho(\lambda) e^{-\beta\lambda} \quad \Rightarrow \quad R(E) = \int_\Omega d\lambda \frac{\rho(\lambda)}{E - \lambda}, \tag{60}$$

where Ω is the support of ρ, also called the spectral domain. It also admits a genus expansion. Denoting by ρ_0 the density at the same order as $R_{0,1}$, one gets

$$R_{0,1}(E) = \int_{\Omega_0} d\lambda \frac{\rho_0(\lambda)}{E - \lambda} \tag{61}$$

Note that in general $\Omega \neq \Omega_0$. $R_{0,1}$ is determined through the $k = G = 0$ loop equation

$$R_{0,1}(E)^2 = V'(E) R_{0,1}(E) - P_{0,0}(E), \tag{62}$$

which is solved by introducing polynomials $Q(E)$ and $\sigma(E)$ such that $V'^2 - 4P_{0,0} = Q^2 \sigma$. Then,

$$R_{0,1} = \frac{1}{2}\big(V' - Q\sqrt{\sigma}\big). \tag{63}$$

Q is determined by the asymptotic condition $R(E) \sim 1/E + \mathcal{O}\big(1/E^2\big)$ at large E. The roots of σ determine the spectral domain Ω_0 through

$$\rho_0(E) = -\frac{1}{2\pi i}[R_{0,1}(E + i\epsilon) - R_{0,1}(E - i\epsilon)]. \tag{64}$$

As $\sqrt{\sigma}$ changes sign when one crosses over a branch cut, one has

$$\rho_0(E) = \frac{1}{2\pi} Q(E) \sqrt{-\sigma(E)}, \quad E \in \{\text{branch cuts of } \sigma\}. \tag{65}$$

Matrix integrals of the type $\Omega_0 = [a, b]$ are said to be one-cut, as $R_{0,1}$ is defined with a single branch cut in the complex E plane. In such cases, $\sigma(E) = (E - a)(E - b)$.

Computing $R_{0,2}$

I will briefly outline the computation of $R_{0,2}$, following [24], as the end result has physical significance. The leading order $k = 1$ loop equation, combined with singular properties of σ and M imply

$$R_{0,2}(E_1 + i\epsilon, E_2) + R_{0,2}(E_1 - i\epsilon, E_2) = -\frac{1}{(E_1 - E_2)^2}, \quad E_1 \in \Omega_0. \tag{66}$$

This expression is unique in the sense that its only dependence on the matrix potential is through the support Ω_0. It is a universal quantity, shared between all matrix models with the same leading order spectral domain.

One then writes down the general solution with the appropriate behaviour when E_1 or E_2 goes through a cut:

$$R_{0,2}(E_1, E_2) = \frac{1}{2(E_1 - E_2)^2}\left(\frac{Q_2(E_1, E_2)}{\sqrt{\sigma(E_1)}\sqrt{\sigma(E_2)}} - 1\right), \tag{67}$$

where $Q_2(E_1, E_2) = Q_2(E_2, E_1)$ is determined through a set of constraints implied by the analytic structure and the asymptotic behaviour of $R_{0,2}$. In the special one-cut case, where $\Omega_0 = [a, b]$, the universal result is

$$R_{0,2}(E_1, E_2) = \frac{1}{2(E_1 - E_2)^2}\left(\frac{E_1E_2 + ab - (a + b)(E_1 + E_2)/2}{\sqrt{\sigma(E_1)}\sqrt{\sigma(E_2)}} - 1\right), \tag{68}$$

which is manifestly independent from $V(M)$ except through the end-points of Ω_0.

3.3 Topological recursion

Systematically combining the set of loop equations with the genus expansion, one ends up with the so-called topological recursion formulation of [24, 25]. Define the Riemann surface Σ immersed in $(\mathbb{C} \cup \infty) \times \mathbb{C}$ by

$$i(\Sigma) = \{(x, y) \in (\mathbb{C} \cup \infty) \times \mathbb{C} \mid y^2 - V'(x)y + P_{0,0}(x) = 0\}, \tag{69}$$

where the equation for the spectral curve y is the loop equation for $R_{0,1}$. Let z denote a coordinate chart on Σ. The points at which the map $x : z \mapsto x(z)$ degenerates are branch points. It is conventional to take $x = -E$.

The local Galois involution $\sigma_a(z)$ about branch point a exchanges the two sheets meeting at a. Operationally, assuming dx has a simple zero at a, this corresponds to exchanging the sign of the square root in the inverse map $z(x)$:

$$z(x) = a + \sqrt{\frac{2}{x''(a)}(x - x(a))} + \cdots \quad \Rightarrow \quad \sigma_a(z(x)) = a - \sqrt{\frac{2}{x''(a)}(x - x(a))} + \cdots \tag{70}$$

The key result is that one can recursively define n-forms $\omega_{G,n}$ which at given order (G, n) satisfy the loop equations. Defining the inputs to the recursion

$$\omega_{0,1} = ydx, \quad \omega_{0,2} = \frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2}, \tag{71}$$

and the recursion kernel around branch point a

$$K_a(z_1,z)=\frac{1}{2}\frac{\int_{\sigma_a(z)}^{z}du\,\omega_{0,2}(z_1,u)}{\omega_{0,1}(z)-\omega_{0,1}(\sigma_a(z))},\tag{72}$$

the remaining $\omega_{G,n}$ are given through

$$\omega_{G,n}(z_1,I)=\sum_a\operatorname*{Res}_{z=z_0}K_a(z_1,z)\left[\omega_{G-1,n+1}(\sigma_a(z),I)+\sum_{h+h'=g}^{\prime}\omega_{h,1+|J|}(z,J)\omega_{h',1+|J'|}(\sigma_a(z),J')\right],\tag{73}$$

where $I=\{z_2,\ldots,z_n\}$, $J\cup J'=I$ and $\sum'$ excludes $(h,J)=(0,\emptyset)$ and $(h,J)=(g,I)$. $\omega_{G,n}$ are related to the connected resolvents through

$$\omega_{G,n}(z_1,\ldots,z_n)dz_1\wedge\cdots\wedge dz_n=R_{G,n}(x_1,\ldots,x_n)dx_1\wedge\cdots\wedge dx_n.\tag{74}$$

3.4 Double scaling

Double scaling is a particular large N limit, where either one or both of the spectral edges are sent to infinity. This procedure defines a one-parameter family of matrix integrals.

More precisely, one takes $N\to\infty$ and tunes the couplings $g_p\to g_p^c$ such that the leading order density of states remains fixed [10]:

$$\rho_0^{\text{total}}(E)=\frac{1}{\hbar}\rho_0(E),\tag{75}$$

where ρ_0^{total} denotes the total spectral density (its integral is N), and $\hbar$ is some finite constant. Intuitively, by fine-tuning the potential appropriately, we are increasing the density of eigenvalues as $N\to\infty$ such that the integral of $\rho_0(E)$ diverges as $N\hbar$. Equivalently, one can think of this procedure as "zooming in" on a part of the spectral density [26].

The formalism of loop equations and topological recursion applies as before, with the only difference being the parameter in the genus expansion $N\to\hbar^{-1}$. For example, now we have a small $\hbar$ expansion for the resolvents

$$\langle R(E_1)\ldots R(E_n)\rangle_c=\sum_{G=0}^{\infty}\hbar^{2G+n-2}R_{G,n}(E_1,\ldots,E_n)+\text{non-perturbative}.\tag{76}$$

The simplification through double scaling is purely due to the fact that in a double scaled model, one does not require $\rho_0(E)$ to be unit normalised. Essentially, we are decompactifying the spectral domain Ω_0.

First, consider the case $\Omega_0=[0,\infty)$. This defines a two-sheet Riemann surface, with global coordinates $z^2=x$. The local Galois involution about $z=0$ is promoted to a global involution $\sigma(z)=-z$. For $y(z)$ odd, the recursion kernel is

$$K_0(z_1,z)=\frac{1}{4y(z)(z_1^2-z^2)},\tag{77}$$

and the recursion relation becomes

$$\omega_{G,n}(z_1,I) = \operatorname*{Res}_{z=0}\left\{\frac{1}{4y(z)(z_1^2-z^2)}\left[\omega_{G-1,n+1}(-z,I) + \sum_{h+h'=g}^{\prime} \omega_{h,1+|J|}(z,J)\omega_{h',1+|J'|}(-z,J')\right]\right\}. \tag{78}$$

This particular limit turns out to be relevant for JT gravity.

Alternatively, one can send both spectral edges to infinity, setting $\Omega_0 = \mathbb{R}$. In this case, it turns out that the residues at $z = \pm\infty$ vanish and the recursion becomes trivial $\omega_{G,n} = 0$ [27]. The only non-zero resolvents are $R_{0,1}$ and $R_{0,2}$. This case is relevant for flat space holography.

3.5 Connection to gravity

Identification of matrix integrals as non-perturbative completions of two-dimensional quantum gravity theories proceeds by matching perturbative observables on both sides. For JT gravity, the map takes a simple form [9]:

$$Z_{\rm JT}(\beta_1,\ldots,\beta_n) = \left\langle \operatorname{Tr} e^{-\beta_1 M}\ldots \operatorname{Tr} e^{-\beta_n M}\right\rangle_{\rm c}. \tag{79}$$

To establish a correspondence, both sides have to match order by order in perturbation theory. The free parameter $\hbar$ introduced in the double scaling limit can be fixed to the genus expansion parameter on the JT side, $\hbar = e^{-S_0}$. One then tunes the matrix potential such that the leading order one-boundary partition functions match. In terms of the spectral density,

$$Z_0^{\rm JT}(\beta) = \int_0^\infty \rho_0^{\rm JT}(E)e^{-\beta E}dE \tag{80}$$

The leading order one-boundary partition function is given by the disk geometry, for which we have (setting $\alpha = 2$):

$$\rho_0^{\rm JT} = \frac{\gamma}{2\pi^2}\sinh\Big(2\pi\sqrt{2\gamma E}\Big). \tag{81}$$

The corresponding spectral curve on the matrix integral side is

$$y(z) = -i\pi\rho_0^{JT}(-z^2) = \frac{\gamma}{2\pi}\sin\Big(2\pi z\sqrt{2\gamma}\Big), \tag{82}$$

where the first equality follows from equation (64). Specifying the double scaling limit and the spectral curve completely determines the matrix integral. All that remains to check is the matching of the rest of the $R_{G,n}$.

$R_{0,2}$ in the one-cut case was computed explicitly in equation (68). In the double-scaled limit, we set $a = 0$ and $b \to \infty$[3], and substituting for $E_i = -z_i^2$ we get

$$R_{0,2}(z_1,z_2) = \frac{1}{4z_1z_2(z_1+z_2)^2}. \tag{83}$$

Under the map (79), this resolvent is related to the two-trumpet JT partition function through equation (47):

$$R_{0,2}(E_1,E_2) = (-1)^2\int_0^\infty d\beta_1 d\beta_2\; e^{\beta_1E_1+\beta_2E_2}Z_{0,2}^{JT}(\beta_1,\beta_2). \tag{84}$$

[3]It doesn't matter the order in which we set $b \to \infty$ and $a \to 0$.

Substituting for $Z_{0,2}^{JT}(\beta_1, \beta_2)$ and performing the Laplace transforms does indeed match with the matrix integral result.

Finally, we check agreement with all higher order $R_{G,n}$. Equivalently, we will check agreement between the topological recursion of (78) with the recursion formula for the Weil-Petersson volumes of bulk moduli surfaces of section 2.4. Equation (74) implies

$$\omega_{G,n}(z_1, \ldots, z_n) = (-2)^n z_1 \ldots z_n R_{G,n}(-z_1^2, \ldots, -z_n^2). \tag{85}$$

Combining with equation (47) yields an expression for $\omega_{G,n}$ in terms of $Z_{G,n}$:

$$\omega_{G,n}(z_1, \ldots, z_n) = 2^n \left(\prod_{i=1}^{n} z_i \int_0^\infty d\beta_i e^{-\beta_i z_i^2} \right) Z_{G,n}(\beta_1, \ldots, \beta_n). \tag{86}$$

Substituting for $Z_{G,n}$ using equations (33) and (38) yields

$$\omega_{G,n}(z_1, \ldots, z_n) = \left(\prod_{i=1}^{n} 2^n z_i \left(\frac{\gamma}{2\pi\beta} \right) \int_0^\infty d\beta_i \int_0^\infty b_i db_i e^{-\gamma b_i^2/2\beta_i} \right) V_{G,n}(b_1, \ldots, b_n). \tag{87}$$

Performing the integral over β_i gives

$$\omega_{G,n}(z_1, \ldots, z_n) = \left(\prod_{i=1}^{n} \sqrt{2\gamma} \int_0^\infty b_i db_i e^{-\sqrt{2\gamma} b_i z_i} \right) V_{G,n}(b_1, \ldots, b_n). \tag{88}$$

It was shown in [28] that Mirzakhani's recursion relation for $V_{G,n}$ implies the topological recursion of equation (78) for $y(z) = \sin(2\pi z)/4\pi^2$ and $\gamma = 1/2$. It is straightforward to check that this correspondence remains valid for arbitrary γ. This completes the correspondence between JT and a double-scaled one-cut matrix integral to all orders in e^{-S_0}.

3.6 Non-perturbative analysis

The non-perturbative analysis of [26, 29, 30, 31, 32] starts by introducing a system of polynomials $P_n(\lambda) = \lambda^n + \mathcal{O}(\lambda^{n-1})$ orthogonal with respect to the matrix potential measure $d\mu(\lambda) = d\lambda e^{-NV(\lambda)}$:

$$(P_n, P_m) \equiv \int_{-\infty}^{\infty} d\mu(\lambda) P_n(\lambda) P_m(\lambda) = h_n \delta_{nm}. \tag{89}$$

These polynomials encode all information about the matrix model, one can express the partition function and all observables in terms of them. Define the related set of orthonormal functions

$$\psi_n(\lambda) = \frac{1}{\sqrt{h_n}} e^{-NV(\lambda)/2} P_n(\lambda) \quad \Rightarrow \quad \int_{-\infty}^{\infty} d\lambda \psi_n(\lambda) \psi_m(\lambda) = \delta_{nm}. \tag{90}$$

At finite N, the joint probability distribution for a subset of eigenvalues $p_n(\lambda_1, \ldots, \lambda_n)$ an be expressed in terms of the ψ_n through the kernel

$$K(\lambda_i, \lambda_j) = \sum_{n=0}^{N-1} \psi_n(\lambda_i) \psi_n(\lambda_j), \tag{91}$$

such that

$$p_n(\lambda_1, \ldots, \lambda_n) = \det \Big[K(\lambda_i, \lambda_j)\Big]_{i,j=1}^{n}. \tag{92}$$

As shown in [26], it follows that the cumulative distribution for the nth eigenvalue to lie in range (a, b) is given by

$$c(n; (a, b)) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \frac{\mathrm{d}^j}{\mathrm{d}z^j} \mathcal{F}[(a, b); z)], \tag{93}$$

where the Fredholm determinant[4] $\mathcal{F}$ is defined in terms of the integral operator

$$\mathbf{K} \,|_{(a,b)} \colon f(\lambda) \to \int_a^b d\kappa \; K(\lambda, \kappa) f(\kappa) \tag{94}$$

as

$$\mathcal{F}[(a, b); z)] = \det\big(1 - z\mathbf{K} \,|_{(a,b)}\big). \tag{95}$$

It encodes the exact spectrum. The challenge is its non-perturbative computation in the double scaling limit.

Double-scaling limit

In two-dimensional gravity literature, two different types of double-scaling limits are often used. One type considers tuning the matrix potential $g_p \to g_p^c$ such that in the large N limit, the partition functions at each genus diverges in a perturbative expansion in the couplings. For an interpretation of this in terms of triangulations of random surfaces, see [33]. The idea is that as higher genus contributions are enhanced, in a correlated limit of $N \to \infty$ and $g_p \to g_p^c$, one obtains finite contributions from all genuses. It is accompanied with zooming into the eigenvalues at the edge of the spectral density. This is the double-scaling limit used in JT gravity literature.

The alternative double-scaling limit considers tuning the potential to a critical point corresponding to a phase transition from one-cut to double-cut phases. The prototypical example of this is the potential

$$V(M; \kappa) = \frac{1}{\kappa}\left(-M^2 + \frac{1}{4}M^4\right), \tag{96}$$

in the limit $\kappa \to 1$. It was shown in [27] that in this limit, zooming into the eigenvalues around zero yields a spectral density that matches with non-supersymmetric CJ gravity.

In either case, one considers a scaling ansatz $1/N = \hbar\delta^{2k+1}$, where $\delta \to 0$ and k is an integer determined by the order of the matrix potential. This is supplemented by a rescaling of the eigenvalues around some value $\lambda = \lambda_c - E\delta^2$, and of the polynomial index $n/N \to 1 - x\delta^{2k}$. It was shown in [29] that the orthogonal functions $\psi_n(\lambda) \to \psi(x, E)$ satisfy the differential equation

$$\left[-\hbar^2 \frac{\partial^2}{\partial x^2} + u(x)\right]\psi(x, E) = E\psi(x, E), \tag{97}$$

with $u(x)$ determined through a non-linear ODE determined by the order of the matrix potential. The rest of the analysis proceed numerically, but the idea is to first solve for $u(x)$, then obtain

[4] This object is a Fredholm determinant, as it is the determinant of the difference between the identity and an integral operator.

$\psi(x, E)$ from which the continuum limit of the kernel in equation (91) can be computed. Then, one calculates the Fredholm determinant and extracts the exact microstate spectrum.

4 Supersymmetric Cangemi-Jackiw gravity

This section summarises the results of [14] on supersymmetric Cangemi-Jackiw (SCJ) gravity.

4.1 BF formulation

Following the analysis of section 2.2, it is useful to start directly with a BF formulation. As before, we introduce the spacetime scalar B and one-form connection A valued on the algebra of a gauge group G. Since we are interested in obtaining a flat space dilaton gravity theory, we choose the gauge algebra $\mathfrak{g}$ to contain the Minkowski (Euclidean) isometries. Hence, introduce as generators a pair of translations P_a and a rotation J obeying

$$[J, P_\pm] = \pm P_\pm. \tag{98}$$

To endow the space with $\mathcal{N} = 1$ supersymmetry, introduce a pair of fermionic generators $Q_\pm$ with

$$[J, Q_\pm] = \pm\frac{1}{2}Q_\pm. \tag{99}$$

The Poincaré algebra does not admit a non-degenerate bilinear form. As shown in [34], this can be remedied by an inclusion of a central element I. This centrally extended superalgebra is referred to as the Maxwell algebra with the additional non-zero (anti)commutators [35]

$$[P_+, P_-] = I, \quad \{Q_+, Q_-\} = I. \tag{100}$$

It turns out to be convenient [14] to break the $\pm$ symmetry and include another pair of non-zero commutators:

$$[P_-, Q_+] = -\frac{1}{2}Q_-, \quad \{Q_+, Q_+\} = P_+. \tag{101}$$

This completes the Maxwell superalgebra. The non-degenerate bilinear form obtained from the quadratic casimir has the non-zero elements

$$\langle P_+, P_-\rangle = 1, \quad \langle J, I\rangle = 1, \quad \langle Q_+, Q_-\rangle = 2. \tag{102}$$

The BF theory action is

$$I_{\text{BF}}[A, B] = \int_{\mathcal{M}} \langle B, F\rangle - \frac{1}{2}\int_{\partial\mathcal{M}} \langle B, A\rangle\,. \tag{103}$$

4.2 Bulk spacetime action

The connection to the gravitational theory is made through identifying fields, zweibeins and the spin connection as coefficients of generators

$$A = \sum_{t=\pm}(e^t P_t + \psi^t Q_t) + \omega J + \tilde{A} I, \tag{104}$$

$$B = \sum_{t=\pm}(x^t P_t + \lambda^t Q_t) + \Psi J + \Phi I. \tag{105}$$

The theory contains two scalar fields Φ, Ψ, a gauge field $\tilde{A}$ and two fermionic fields $\psi^{\pm}$. The pairs $x^{\pm}$ and $\lambda^{\pm}$ are non-dynamical and don't appear in the bulk bosonic action, which reads

$$I_{\rm SCJ}\Big|_{\lambda,\psi=0} = \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{|g|}\Big[\Phi R + 2\Psi(\varepsilon^{\mu\nu}\partial_\mu \tilde{A}_\nu + 1)\Big]. \tag{106}$$

Clearly, the dilaton Φ fixes $R = 0$, so the on-shell geometry is locally flat. This action is supplemented by a boundary term derived from the BF action.

The on-shell solution used in [14] is first written in Bondi gauge for Lorentzian signature, then analytically continued to Euclidean time. This yields complex zweibeins and spin connection

$$e^+ = i(P(\tau)r + T(\tau))d\tau - dr, \quad e^- = -id\tau, \quad \omega = -iP(\tau)d\tau, \tag{107}$$

where $(P,T) = (2\pi/\beta, 0)$ for the disk and $(P,T) = (0, b^2/2\beta^2)$ for the cylinder with circumference b. These complex valued saddles are consistent with the Konstevich-Segal criteria of [36, 37]. It would be interesting to explore the implications of complex saddles further in this context, specifically asking whether there are more complex saddles that can appear as off-shell contributions.

Supersymmetry transformations are realized as a subgroup of BF gauge transformations. Parametrising by Grassman functions $\epsilon^{\pm}$, they are induced by

$$\Theta = \epsilon^+ Q_+ + \epsilon^- Q_-, \quad G = 1 + \Theta. \tag{108}$$

Invariance under such supersymmetry transformations are implied by the gauge invariance of the BF formulation.

4.3 Euclidean partition function

Consider the Euclidean partition function for n asymptotic boundaries of lengths β_i. The boundary conditions are such that bosonic fields are periodic, and fermionic fields are antiperiodic around boundary circles. The partition function is

$$Z(\beta_1,\dots,\beta_n) = \int [dX] e^{-I_{\rm SCJ}[X] + S_0\chi(\mathcal{M})}, \tag{109}$$

where the addition of the Euler characteristic term is ad-hoc, unlike JT gravity where it arises naturally from dimensional reduction. The integral over the metric instructs us to sum over all allowable Euclidean manifolds $\mathcal{M}$, leading to the usual genus expansion

$$Z(\beta_1,\dots,\beta_n) = \sum_{g=0}^{\infty}(e^{-S_0})^{2(g-1)+n} Z_g(\beta_1,\dots,\beta_n) + \text{non-perturbative}. \tag{110}$$

Noting that $I_{\rm SCJ} \supset \int \Phi d\omega$, taking an integration contour for Φ along a purely imaginary line gives a factor of $\delta(R)$.

It is known that there are only two choices of elliptic compact Riemann surfaces: the disk and the cylinder [38]. Hence, the genus expansion simplifies dramatically:

$$Z(\beta) = e^{S_0} Z_{\text{disk}}(\beta) + \text{non-perturbative} \tag{111}$$

$$Z(\beta_1, \beta_2) = Z_{\text{cylinder}}(\beta_1, \beta_2) + \text{non-perturbative} \tag{112}$$

$$Z(\beta_1, \ldots, \beta_n) = \text{non-perturbative for } n > 2. \tag{113}$$

The computation of the disk and cylinder partition functions proceed in a similar manner to the JT case, one obtains induced measures from the BF formulation and imposes appropriate asymptotic boundary conditions to determine allowed large gauge transformations. As before, the Duistermaat-Heckman theorem holds and the resulting integrals are one-loop exact.

The resulting partition functions with Zeta function regularization read

$$Z_{\text{disk}}(\beta) = \frac{2\sqrt{2}}{\gamma\beta}, \quad Z_{\text{cylinder}}(\beta_1, \beta_2) = \frac{1}{\gamma(\beta_1 + \beta_2)}, \tag{114}$$

where the arbitrary constant γ is inherited from the BF theory boundary condition. Noting that both partition functions are functions only of the combination $\gamma\beta_i$, wlog one can fix $\gamma = 1$. This amounts to a choice of units.

4.4 Matrix model completion

Following the analysis of section 3.4, the relevant double-scaling limit is one which sets $\Omega_0 = \mathbb{R}$ so that only $R_{0,1}$ and $R_{0,2}$ remain non-zero. All that remains is identifying the map under which $R_{0,1}$ and $R_{0,2}$ yield the disk and cylinder partition functions.

In the case of JT gravity, the connection to the matrix integral was made through the map given by equation (79). Equivalently, the matrix operator that corresponded to the insertion of an asymptotic boundary of length β was $\operatorname{Tr} e^{-\beta M}$. In that case, the random matrices M acquire an interpretation as boundary Hamiltonians. We generalize this notion to consider maps of the form

$$Z_{\text{SCJ}}(\beta_1, \ldots, \beta_n) = \langle \mathcal{O}(\beta_1) \ldots \mathcal{O}(\beta_n) \rangle_c \,, \tag{115}$$

where the operators $\mathcal{O}(\beta_i)$, corresponding to the insertion of an asymptotic boundary, are to be determined.

At this point, the universality of $R_{0,2}$ becomes crucial. Taking the limit of (68) as $a = b \to \infty$ with a branch cut along the real positive line yields

$$R_{0,2}(E_1, E_2) = \frac{-1}{(E_1 - E_2)^2} \tag{116}$$

if E_1 and E_2 are on different sheets of the Riemann surface, and zero otherwise. The map $\mathcal{O}(\beta)$ can be determined through matching with the cylinder partition function. It has the form

$$\mathcal{O}(\beta) = \int_{-\infty}^{\infty} dp \ \operatorname{Tr}\left[\exp\bigl(-\beta(M^2 + p^2)\bigr)\right]. \tag{117}$$

The integral over p is a feature of flat space gravity, also appearing in the non-supersymmetric case

[27, 13]. With this choice of $\mathcal{O}(\beta)$, one then tunes the matrix potential such that

$$\langle \mathcal{O}(\beta) \rangle \simeq e_0^S Z_{\rm disk}(\beta) = e_0^S \frac{2\sqrt{2}}{\beta}. \tag{118}$$

After performing the integral over p, the left hand side can be expressed in terms of the leading order spectral density:

$$\langle \mathcal{O}(\beta) \rangle = \sqrt{\frac{\pi}{\beta}} \int_{-\infty}^{\infty} d\lambda \, e^{-\beta\lambda^2} \rho_0(\lambda). \tag{119}$$

The choice of ρ_0 that matches with the disk is simply the constant density $\rho_0 = 1/\pi\hbar$ where $\hbar = e^{-S_0}/2\sqrt{2}$. This can be obtained from the Gaussian potential $V(M) = \frac{1}{2}M$ in the double-scaled limit $\hbar^{-1} = \delta N$ and $\lambda = \delta E$. This is not a conventional double-scaling limit, as there are no couplings in the potential to tune to criticality. We elaborate on this point in section 5.5. As shown in [14], this matrix model yields the following analytic solution for the kernel

$$K(E, E') = \frac{1}{\pi} \frac{\sin((E - E')/\hbar)}{E - E'}, \tag{120}$$

from which non-perturbative observables of the gravitational theory can be obtained, an example of which is the exact spectral density:

$$\rho(E) = \lim_{E \to E'} K(E, E') = \frac{1}{\pi\hbar}. \tag{121}$$

This matches exactly with the leading order density, suggesting that $\langle \rho(E) \rangle$ does not receive any non-perturbative corrections. Multi-boundary observables, however, do receive non-perturbative corrections. The spectral form factor $S(\beta, t)$, defined in the bulk theory as

$$S(\beta, t) = Z(\beta + it)Z(\beta - it) + Z(\beta + it, \beta - it), \tag{122}$$

is a commonly studied quantity in the two-dimensional gravity literature. For SCJ gravity, it takes the exact analytic form

$$S(\beta, t) = \frac{1}{\hbar^2} \frac{1}{\beta^2 + t^2} + \frac{1}{2\beta}\Big(1 - e^{-f(\beta,t)}\Big) + \frac{1}{2\beta}\sqrt{\frac{\pi}{f(\beta,t)}}\Big[1 - \text{Erf}\Big(\sqrt{f(\beta,t)}\Big)\Big], \tag{123}$$

where $f(\beta, t) \equiv 2\beta\hbar^{-2}/(\beta^2 + t^2)$.

5 Future directions

5.1 The factorisation puzzle

The Euclidean gravitational path integral, defined as a sum over all spacetime manifolds, presents an immediate challenge in the context of holography. Multi-boundary partition functions and observables do not factorise, in the sense $Z_{12} \neq Z_1 Z_2$, due to so-called wormhole contributions connecting the boundaries. This presents a challenge for holography as it implies that the holographic duals cannot factorise either [39]. Results identifying matrix duals to two-dimensional theories [9, 10, 11, 27, 14] have provided examples of non-factorising boundary theories, hence raising the question of whether gravity is fundamentally an ensemble.

A possible resolution is provided through arguing the gravitational path integral shouldn't sum over topologies, removing the source of the problem [40]. However, further evidence for the inclusion of wormhole geometries is provided through the "replica wormhole" computations of [4, 5], recovering the Page curve for evaporating black holes.

An alternative solution presented in [26] identifies the underlying microstate spectrum of the matrix dual to JT gravity with a discrete spectrum for JT gravity. In other words, they propose that the Euclidean path integral captures contributions from an ensemble of deformations of JT gravity, thereby producing a smooth spectrum to an otherwise underlying discrete one.

It would be interesting to extend the analysis of [26] to extract the discrete spectra associated with a class of deformations to JT gravity. Matrix duals to a large set of deformations were already found in [11], so this task is certainly possible. After the discrete spectra of deformations are found, one can check whether a sum of them indeed produces the smooth spectrum of Euclidean JT gravity, hence providing a check on the proposed solution of [26] to the factorisation puzzle.

5.2 Two-dimensional swampland

In string theory literature, swampland is a term that refers to the space of consistent quantum field theories that, from the string theory perspective, can not be consistently coupled to gravity [41]. The complement of the swampland is called the landscape, and it is known that the landscape is measure zero with respect to the swampland [41]. Conjecturing and proving criteria that identifies elements of the swampland is a difficult task given the huge number of string compactifications.

One can instead consider a similar classification scheme in the space of two-dimensional theories, where the criteria of UV completeness is provided by the existence of a matrix dual rather than a string compactification. Since we know for certain that theories with matrix duals are UV complete, this criterion is guaranteed to yield a subset of the two-dimensional landscape. However, we can't make any statements about the complement, that is to say there is no proof that theories without matrix duals are UV incomplete.

As a future research direction, one can consider the universality of $R_{0,2}$ as a first step in constraining the 2d swampland. This resolvent puts a direct constraint on the genus zero two-boundary amplitude, given the leading order spectral domain of the theory which is provided by the genus zero one-boundary amplitude. It would be interesting to see whether this implies restrictions on the allowed matter sector or dilaton deformations of, say JT gravity. The main limitation of such an approach would be the ability to perform calculations on the gravitational side. It has been shown that each element in a class of JT deformations of the form

$$I = -\frac{1}{2}\int \sqrt{g}(\phi R + W(\phi)) \tag{124}$$

has a matrix dual [11]. Given the amount of theoretical control two-dimensional theories provide, further progress can be made in this line of investigation.

5.3 Non-perturbative structure of SCJ gravity

The proposed matrix dual to SCJ gravity admits an analytic non-perturbative solution. This allows a precise investigation of the structure of non-perturbative corrections. As an example, consider the spectral form factor of equation (123) in the semiclassical $\hbar \to 0$ limit. The complementary error function admits an asymptotic expansion for large x of the form

$$\mathrm{Erfc}(x) \equiv 1 - \mathrm{Erf}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2x^2)^n}. \tag{125}$$

This yields an asymptotic expansion for the spectral form factor for small $\hbar$:

$$S(\beta,t) = \frac{1}{\hbar^2}\frac{1}{\beta^2+t^2} + \frac{1}{2\beta}\left(1 + \exp\left[\frac{-2\beta}{\hbar^2(\beta^2+t^2)}\right]\sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!!}{2^n}\left(\hbar^2\frac{\beta^2+t^2}{2\beta}\right)^n\right). \tag{126}$$

In this form, it is clear that only the connected two-boundary term receives non-perturbative corrections. This is expected since the single boundary observables are non-perturbatively exact. Recall, from the genus expansion in equation (109), a genus g manifold with two boundaries is accompanied with a factor of $\hbar^{2g}$. From this perspective, it might possible to endow the corrections above with a genus expansion interpretation. The only caveat is the $\exp\big(-2\beta\hbar^{-2}/(\beta^2+t^2)\big)$ term multiplying the series. Furthermore, the non-perturbative correction is not a function of β, t and $\hbar$ independently, but only of the combination $\hbar^2(\beta^2+t^2)/2\beta$. This is nothing but the ratio of the cylinder and disk partition functions.

As a future direction, it might be worthwhile to investigate whether there is an underlying geometrical interpretation to the non-perturbative corrections. The problem essentially reduces to solving the bulk theory exactly. There is no obvious starting point, but a detailed study of complex saddle point geometries might be a possible avenue [37]. It should be noted that if one is able to find a procedure that recovers these non-perturbative corrections directly from the gravitational path integral, one would then be able to exactly define the bulk theory without needing a matrix model completion. Moreover, generalisations to other two-dimensional (and perhaps higher dimensional) theories might be possible.

5.4 Free particle sector of SCJ gravity

The matrix operator $\mathcal{O}(\beta)$, corresponding to the insertion of an asymptotic boundary of length β on the bulk, allows one to define an effective Hamiltonian H, called the Bondi Hamiltonian, through

$$\mathcal{O}(\beta) = \int_{-\infty}^{\infty} dp\ \mathrm{Tr}\left[\exp\big(-\beta(M^2+p^2)\big)\right] \equiv \mathrm{Tr}\, e^{-\beta H}. \tag{127}$$

The Bondi Hamiltonian receives contributions from the matrix model sector through M^2, and a free particle sector through p. The authors of [27] suggest that the free particle sector can be traced back to the central extension of the Poincaré algebra, but state that its physical origin is unclear as it doesn't appear at the level of the bulk action.

In this section, I would like to note that the volume of the centre of the algebra does appear in the Fadeev-Popov-BRST procedure outlined in appendix 1. Schematically, the path integral measure would read

$$d\mu[\tau] = \frac{1}{\mathrm{Vol}\, Z(G)} \lim_{N\to\infty} \frac{1}{N!}(\Omega[d\varepsilon])^N, \tag{128}$$

where $Z(G)$ is the centre of the Maxwell algebra. Perhaps a careful analysis of this centre contribution can shed light on the appearance of the free particle sector.

5.5 Uniqueness of the SCJ matrix dual

The authors of [14] claim that a constant spectral density is obtained from a double-scaled Gaussian potential $V(M) = \frac{1}{2}M^2$. However, I would like to point out that it is not obvious that this is the unique solution. One can tune any even potential with $\rho_0(0) > 0$ such that in a limit zooming in on $\lambda \sim 0$ the resulting double-scaled spectral density is constant. As an example, consider the quartic potential

$$V(M) = g_2 M^2 + g_4 M^4, \tag{129}$$

For $g_2 > 0$, the leading order spectral density for this model has the following expression

$$\rho_0(\lambda) = \frac{1}{\pi}\,\mathrm{Re}\int_0^1 \frac{dX}{\sqrt{4R(X) - \lambda^2}}, \tag{130}$$

where

$$R(X) = \frac{g_2}{12 g_4}\left[\sqrt{1 + \frac{12 g_4}{g_2^2}X} - 1\right]. \tag{131}$$

A detailed derivation of this in terms of recursion coefficients is provided in appendix 3. One can then consider a double-scaling limit with $X = x\delta^2$, $\lambda = E\delta$ and $\hbar^{-1} = N\delta$. The resulting spectral density is

$$\rho_0(E) = \frac{1}{\pi\hbar}\lim_{\delta\to 0}\delta^2 \int_0^{1/\delta^2} \frac{dx}{\sqrt{4R(x\delta^2) - \delta^2 E^2}}. \tag{132}$$

This is plotted as a function of E and g_4 in figure 1, and indeed takes on a constant value depending on (g_2, g_4).

One might argue that in a double-scaling limit as simple as this, it is plausible that a large set of matrix models fall into the same universality class, thereby can be identified. A proof of this claim would need to show that, under the appropriate double-scaling limit, all such models reproduce the same non-perturbative behaviour. In other words, the orthogonal functions $\psi_n(E)$ for all such models obey the same differential equation. However, there is some evidence against this claim. The expression for $\rho_0(E)$ is well defined for all $g_4 > -g_2^2/12$, which includes a range of negative g_4 values. In such cases, one expects the matrix potential to be non-perturbatively unstable. This was discussed briefly for the quartic potential in [26], from the point of view of a different double-scaling limit, but should also hold in this case as the matrix potential is not bounded below for $g_4 < 0$. Such

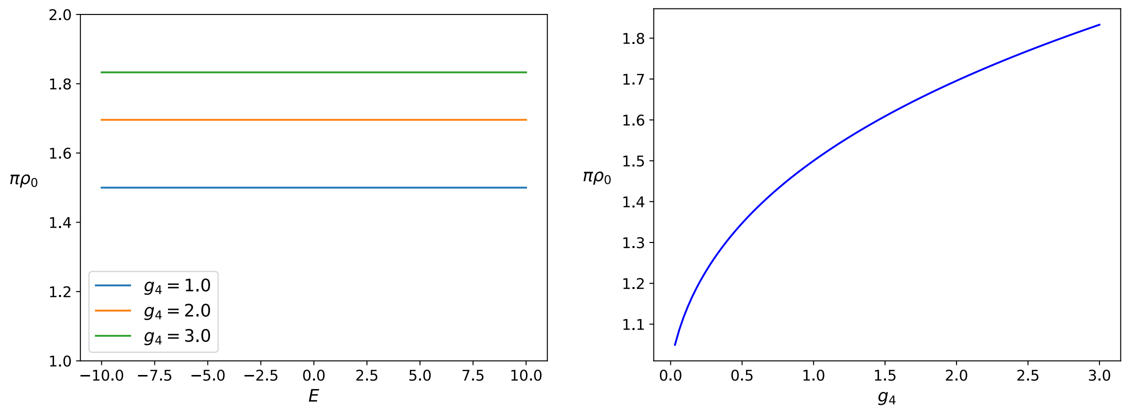

Figure 1: (Left) leading order spectral density $\pi\rho_0(E)$ as a function of E, for $g_4 \in \{1, 2, 3\}$. (Right) $\pi\rho_0(E)$ as a function of g_4 for fixed E. For both figures, $g_2 = 1/2$ and $\delta = 10^{-6}$.

matrix models clearly don't belong to the same universality class as the non-perturbatively stable Gaussian model.

I think a resolution of this tension can be provided by a more careful definition and analysis of the double-scaling limit taken in [14]. As mentioned before, this is not a conventional double-scaling limit as the couplings are not tuned to critical values, and indeed the matrix model itself is not at a critical point.

Appendix A

Supplementary material

1 Faddeev-Popov-BRST procedure

Following [17], the formal definition of a path integral for the BF theory proceeds by quotienting by the volume of the group of maps $\hat{G}:\mathcal{M}\to G$. Let $[dA']$ denote the gauge fixed measure. First, note that the subgroup of gauge transformations that leave an arbitrary A invariant are constant maps from $\mathcal{M}$ to the centre $Z(G)$. These elements cannot be fixed by the usual Fadeev-Popov-BRST procedure and so one quotients them by hand. The path integral reads

$$\frac{1}{\mathrm{Vol}\,\hat{G}}\int [dA][dB]e^{-I_{\mathrm{BF}}}=\frac{1}{\mathrm{Vol}\,Z(G)}\int [dA'][dB]e^{-I_{\mathrm{BF}}}. \tag{A.1}$$

Since the centre of $SL(2,\mathbb{R})$ consists of two elements $\{I,-I\}=\mathbb{Z}_2$, this contributes a factor of $1/2$ to the path integral. In literature, this centre factor is omitted as it can be absorbed into the normalization of the measure.[1]

Next, one imposes a gauge fixing condition. The covariant derivative D acts on a p-form η as $D\eta\equiv d\eta+[A,\eta]$ where the wedge products in the commutator are implicit. Let $A=A^{(0)}+\beta$ for some fixed $A^{(0)}$, and choose the following Lorentz-like gauge condition:

$$G[A]=D_i^{(0)}B^i=0, \tag{A.2}$$

where $D_i^{(0)}$ is the covariant derivative about $A^{(0)}$. Note that this gauge condition relies implicitly on a choice of metric on $\mathcal{M}$. These are a set of conditions, one for each group generator, so it is exhaustive.

BRST procedure follows by defining ghost and anti-ghost fields c and $\bar{c}$, which are anti-commuting zero-forms, along with a commuting zero-form w, all transforming in the adjoint. BRST transformations are

$$\delta A=-Dc,\quad \delta c=\frac{1}{2}[c,c],\quad \delta\bar{c}=iw,\quad \delta w=0. \tag{A.3}$$

[1] I am not aware of additional complications in cases where $Z(G)$ is not discrete, one would proceed by defining an appropriate measure to compute $\mathrm{Vol}\,Z(G)$.

The gauge fixing term in the action is

$$I_{GF} = \int_{\mathcal{M}} \sqrt{g}\Big[\, \langle iw, D_i^{(0)} B^i\rangle + \langle D_i^{(0)}\overline{c}, D^i c\rangle\Big], \tag{A.4}$$

and so the gauge-fixed partition function reads

$$\begin{aligned} Z_{BF} =& \frac{1}{\mathrm{Vol}\, Z(G)} \int [dA][dB][dc][d\overline{c}][dw] \\ & \times \exp\bigg(-I_{BF} - i \int_{\mathcal{M}} \sqrt{g}\, \langle w, D_i^0 B^i\rangle - \int_{\mathcal{M}} \sqrt{g}\, \langle D_i^{(0)}\overline{c}, D^i c\rangle\bigg). \end{aligned} \tag{A.5}$$

The key result follows from precisely defining measures on the spaces $\mathcal{A}$ of connections and $\mathcal{B}$ of $\mathfrak{g}$ valued zero-forms on $\mathcal{M}$. The space $\mathcal{A}$ has a natural measure induced from the symplectic form. The tangent space $T_{A^{(0)}}\mathcal{A}$ at point $A^{(0)}$ is spanned by $\mathfrak{g}$ valued one-forms, for which a symplectic structure can be defined as:

$$\Omega(\eta, \omega) = \alpha \int_{\mathcal{M}} \langle \eta \wedge \omega\rangle \equiv \alpha \int_{\mathcal{M}} \langle \eta_a, \omega_b\rangle\, dx^a \wedge dx^b, \tag{A.6}$$

for some constant α. This is independent from the metric on $\mathcal{M}$. For zero-forms however, $\mathcal{M}$ needs to be endowed with a metric g which induces a metric on $\mathcal{B}$:

$$(\lambda, \phi) = \alpha \int_{\mathcal{M}} \sqrt{g}\, \langle \lambda, \phi\rangle\, . \tag{A.7}$$

For compact groups, the metric on $\mathcal{M}$ also induces a metric on $\mathcal{A}$:

$$(\eta, \omega) = \alpha \int_{\mathcal{M}} \langle \eta \wedge \star\omega\rangle\, , \tag{A.8}$$

where $\star$ is the Hodge star operator. However, this metric is not positive-definite for non-compact groups due to the presence of negative eigenvalues in the Killing form [42]. For example, the Killing form of the non-compact group $SL(2,\mathbb{R})$, in the basis of generators defined in section 2.2 is

$$K = 4\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{A.9}$$

which has one negative eigenvalue. This problem can be remedied by defining an operator T which reverses the sign of the negative component, and defining the metric [42]

$$(\eta, \omega) = \alpha \int_{\mathcal{M}} \langle \eta \wedge \star T\omega\rangle\, . \tag{A.10}$$

This metric is manifestly Kahler-compatible with the symplectic form, as $(\eta, \omega) = \Omega(\eta, \star T\omega)$ noting that $(\star T)^2 = -1$. Kahler-compatibility ensures that one can use the symplectic form to define the measure on $\mathcal{A}$ without referencing the metric g on $\mathcal{M}$ [43], while a choice of g remains necessary for defining the measure on $\mathcal{B}$.

Although the path integral seems to depend on g through the zero-form measure, the Jacobian factor associated with change in g cancels out if there are an equal number of bosonic and fermionic zero-form fields, which is the case in this theory. Relatedly, this makes supersymmetric extensions to the bulk theory possible to study following a similar topological gauge theory formulation.

Integrals over $[dB]$ and $[dw]$ give factors of $\delta(F)$ and $\delta(D_i^{(0)}B^i)$, localising the path integral over flat connections. For $\dim \mathcal{A} = 2n$, the volume form is $\mu = \Omega^n/n!$. In the path integral, one takes $n \to \infty$ after appropriate regularization. Finally, one of the results of [17] is that the integrals over $[dc]$ and $[d\bar{c}]$ on orientable $\mathcal{M}$ is simply 1.

After gauge fixing, the BF theory path integral reduces to an integral over flat connections with the measure induced by the symplectic form, and an overall correction factor of $1/\mathrm{Vol}\, Z(G)$.

2 Trousers decomposition

The following is a simple derivation of the number k of closed loops needed to decompose a genus g surface with n boundaries into trousers geometries.

Consider a (g, n) surface with $g > 0$. Any such surface can be written as a connected sum $\Sigma \# T^2$, where Σ is a $(g-1, n)$ surface. To introduce a new boundary, one can cut open the T^2 and glue the two open ends of the torus to two open ends of a trousers geometry. This yields $T^2 \to T^2 \# D^2$, where the disk is defined with a boundary $\partial D^2 = S^1$. Hence the new surface $\Sigma \# T^2 \# D^2$ is of type $(g, n+1)$. Cutting the torus removes one closed geodesic, but gluing with the trousers adds two closed geodesics. The end result is the addition of a closed geodesic, hence we have

$$k(g, n+1) = k(g, n) + 1. \tag{A.11}$$

In the zero genus case, one necessarily has $n \geq 3$ boundaries. To add a new boundary, one simlpy glues one end of a trousers to a boundary, hence again introduces one closed geodesic. This implies

$$k(g, n) = k(g) + n. \tag{A.12}$$

Now, consider adding a genus. One again cuts open the T^2, but to each open end glues one end of two different trousers geometries. Then, the pair of trousers are glued together. This yields $T^2 \to T^2 \# T^2$, hence $g \to g + 1$. The process had one cut and four gluings, so the number of closed geodesics has increased by 3. Hence,

$$k(g+1) = k(g) + 3. \tag{A.13}$$

In the special $g = 0$ case, note again that $n \geq 3$, so one can glue a pair of trousers to each boundary, then glue one pair of boundaries of the trousers. This induces $D^2 \# D^2 \to T^2 \# D^2 \# D^2$, again introducing 3 new closed curves.

Considering the initial condition of a single trousers geometry with $g = 0, k = 0$ and $n = 3$, one gets

$$k(g, n) = 3g + n - 3. \tag{A.14}$$

3 Spectral density through recursion coefficients

The most general recursion relation consistent with the orthogonality condition (89) is a three term recursion of the form

$$\lambda P_n = P_{n+1} + R_n P_{n-1}. \tag{A.15}$$

Considering $(P_{n-1}, dP_n/d\lambda)$ and integrating by parts yields

$$(V' P_n, P_{n-1}) = \frac{n}{N} h_{n-1}, \tag{A.16}$$

which, after substituting for $V'(\lambda)$ and using the recursion relation yields a difference equation for R_n, the order of which is determined by the order of the potential. For the quartic potential, one obtains a quadratic difference equation:

$$R_n[2g_2 + 4g_4(R_{n+1} + R_n + R_{n-1})] = \frac{n}{N}, \tag{A.17}$$

which should be supplemented with the initial conditions $R_0 = 0$ and $R_1 = \int \lambda^2 d\mu(\lambda) / \int d\mu(\lambda)$, following from the relation $R_n = h_n/h_{n-1}$.

In the large N limit, one obtains an infinite number of polynomials labeled by a continuous index $n/N = X \in [0, 1)$. The spectral density takes a simple form in terms of $R_n \to R(X)$ [44]:

$$\rho(\lambda) = \frac{1}{\pi} \operatorname{Re} \int_0^1 \frac{dX}{\sqrt{4R(X) - \lambda^2}}, \tag{A.18}$$

Equation (A.17) becomes a differential equation

$$R(X)\left(2g_2 + 4g_4\left[3R(X) + \frac{1}{N^2}\frac{\partial R(X)}{\partial X^2} + \mathcal{O}\big(1/N^4\big)\right]\right) = X. \tag{A.19}$$

Solving this order by order in $\mathcal{O}\big(1/N^2\big)$ is equivalent to the perturbative analysis of loop equations. In the planar limit one obtains

$$R_0(X) = \frac{1}{12g_4}\left[-g_2 + \sqrt{g_2^2 + 12g_4 X}\right], \tag{A.20}$$

where the positive square root branch is fixed by the boundary condition $R(0) = 0$.

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