The Physics of Bose-Einstein Condensation

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This is a scientific article submitted as a year 2 assignment at Imperial College London.

Introduction

In 1924 and 1925, three papers published by Albert Einstein changed the history of quantum statistics forever. The papers were published under the title "Quantum Theory of a Monatomic Ideal Gas" to the Royal Prussian Academy of Sciences. The first two papers introduced what is today known as *Bose-Einstein statistics*. [1] In his second paper, under a section titled "The Saturated Ideal Gas[1]", Einstein makes a prediction of a phenomenon which will lead to a Nobel Prize in Physics being awarded in 2001 to its experimental observation. This phenomenon was *Bose-Einstein condensation*.

The inspiration for Einstein came from Satyendra Nath Bose. In a letter adressed to Einstein, Bose asked him to translate one of his articles titled "Planck's Law and the Hypothesis of Light Quanta", where he treated light as indistinguishable particles - photons - and took a statistical approach to deriving Planck's law. The paper was translated by Einstein and published in *Zeitschrift für Physik* under Bose's name. [2] Einstein later applied a similar treatment to ideal gases of integer spin.

To put things into perspective, the development of Bose-Einstein statistics took place when quantum mechanics was still in its infancy. Significant development in the theory was made through 1925 to 1927, hence Bose-Einstein statistics actually predates most of the formal quantum theory. Nevertheless, for gases of atoms with integer spin, the theory holds. This shows, in my opinion, the effectiveness of a statistical approach where the number of assumptions made are minimal.

Although the effects of Bose-Einstein condensation were observed as early as 1938 in liquid helium-4 cooled below 2.17 K, the first gaseous condensate which can accurately be described by theory was created and observed by Eric A. Cornell, Wolfgang Ketterle and Carl E. Wieman in dilute gases of alkali atoms. As a result of this, they have jointly won the Nobel Prize in Physics in 2001, 76 years after the theoretical prediction. [3]

This article aims to introduce the reader to the core ideas of Bose-Einstein condensates in a reasonably motivated approach. In the following sections, quantum statistics for bosons and fermions are introduced. More details on the criteria for Bose-Einstein condensation and their properties are presented. Afterwards, various experiments on superconductivity, superfluidity and Bose-Einstein condensation of dilute gases are discussed.

Quantum Statistics

Before focusing on the theory of Bose-Einstein condensation, we must first state some of key results of statistical physics. The scales we concern ourselves with when looking at Bose-Einstein condensation require a quantum mechanical treatment, and we start with the two types of elementary particles: bosons and fermions.

Bosons and Fermions

Our universe is *entirely* made up of either bosonic or fermionic particles. Although the two behave similarly on macroscopic scales (e.g. dilute gases at room temperature), they behave extremely differently on quantum scales. We can define bosons as particles with integer spin, and fermions as particles with half-integer spin.[4] Fermions constitute *matter* at its fundamental level, all quarks and leptons have half-integer spin. Bosons, on the other hand, are quite different at the most fundamental level as they are *force carriers* such as photons, W and Z bosons. This does not mean, however, that all atoms are fermions since a combination of half-integer spin particles can result in an integer spin one. One example of such composite particles is helium-4, which exhibits bosonic properties.

[1] Originally in German: *"Das gesättigte ideale Gas."*

But how does the spin of a particle affect its behaviour in a system? What is the connection between *spin* and *statistics*? Suppose we have a two particle system described by a wavefunction of the form

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \psi_a(\mathbf{r}_1, t)\psi_b(\mathbf{r}_2, t), \tag{1}$$

where particle 1 is in state described by ψ_a and particle 2 in state ψ_b. This system represents two *distinguishable* particles, as otherwise it would not make sense to state that either of the particles was in one of the two states. Examples of such systems are well-localised atoms on a lattice, or systems of non-identical particles.

If we instead impose the condition that the particles are *indistinguishable*, this must be reflected in the wavefunction of the system. The probability density associated with the wavefunction should remain unaffected when we switch $\mathbf{r}_1$ and $\mathbf{r}_2$ around. As it turns out, there are two ways to construct such a wavefunction by slightly modifying equation (1) as follows:

$$\Psi_\pm(\mathbf{r}_1, \mathbf{r}_2, t) = \frac{1}{\sqrt{2}} \left\{ \psi_a(\mathbf{r}_1, t)\psi_b(\mathbf{r}_2, t) \pm \psi_a(\mathbf{r}_2, t)\psi_b(\mathbf{r}_1, t) \right\}. \tag{2}$$

We take it as an axiom that bosons are described by the plus sign and fermions by the minus sign, however it can be proven through relativistic quantum theory. [4]

To see a visual difference between the two cases, refer to Figure 1, where the wavefunctions for a fermionic and bosonic two particle system are plotted. The potential for both cases is identical (one dimensional infinite square well), and they are plotted on the same scale. The horizontal axes correspond to $\mathbf{r}_1$ and $\mathbf{r}_2$ and the vertical axis corresponds to the wavefunction. It seems like fermions *repel* each other, whereas bosons tend to *come together*. This is a direct consequence of (2).

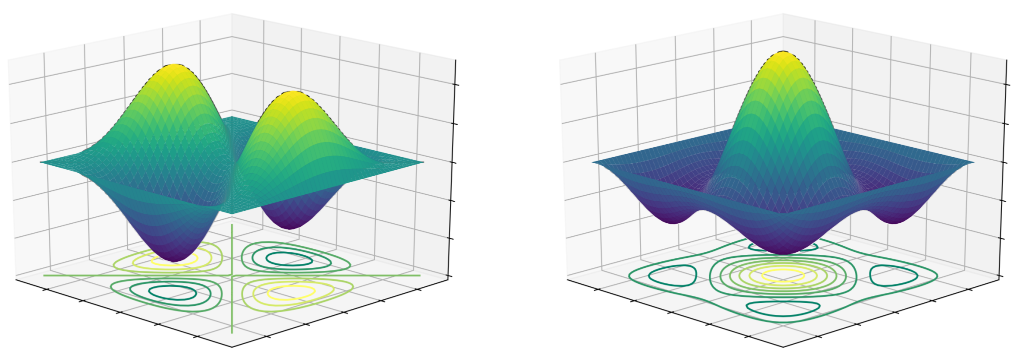

Figure 1: On the left is an asymmetric wavefunction of a fermionic 2-particle state in an infinite square well. On the right is a symmetric wavefunction of a bosonic 2-particle state in the same potential. In both systems, one particle is in the $n = 1$ and the other in the $n = 3$ energy eigenstate. (Created by me, inspired by: Timothy Rias - Wikimedia Commons.)

We see, from equation (2), that if two fermions occupy the same state, the resulting wavefunction is not physical. Letting $\psi_a = \psi_b$, we get

$$\Psi_-(\mathbf{r}_1, \mathbf{r}_2, t) = \frac{1}{\sqrt{2}} \left\{ \psi_a(\mathbf{r}_1, t)\psi_a(\mathbf{r}_2, t) - \psi_a(\mathbf{r}_2, t)\psi_a(\mathbf{r}_1, t) \right\} = 0.$$

This results in the Pauli exclusion principle: two identical fermions cannot occupy the same state.

On the other hand, bosons are *more likely* to occupy the same states. Letting $\psi_a = \psi_b$ on equation (2) we find that

$$\Psi_+(\mathbf{r}_1, \mathbf{r}_2, t) = \frac{1}{\sqrt{2}}\{\psi_a(\mathbf{r}_1, t)\psi_a(\mathbf{r}_2, t) + \psi_a(\mathbf{r}_2, t)\psi_a(\mathbf{r}_1, t)\} = \sqrt{2}\psi_a(\mathbf{r}_1, t)\psi_a(\mathbf{r}_2, t),$$

which results in the probability density

$$P(\mathbf{r}_1, \mathbf{r}_2, t) = 2|\psi_a(\mathbf{r}_1, t)\psi_a(\mathbf{r}_2, t)|^2.$$

The identical calculation using equation (1) yields

$$P(\mathbf{r}_1, \mathbf{r}_2, t) = |\psi_a(\mathbf{r}_1, t)\psi_a(\mathbf{r}_2, t)|^2.$$

Hence, indistinguishable bosons are more likely to occupy the same state. This generalises to n particles as follows: *"if there are already n bosons in a quantum state, the probability of one more joining them is larger by a factor of $(1 + n)$ than it would be if there were no quantum mechanical indistinguishability requirements."* [5]

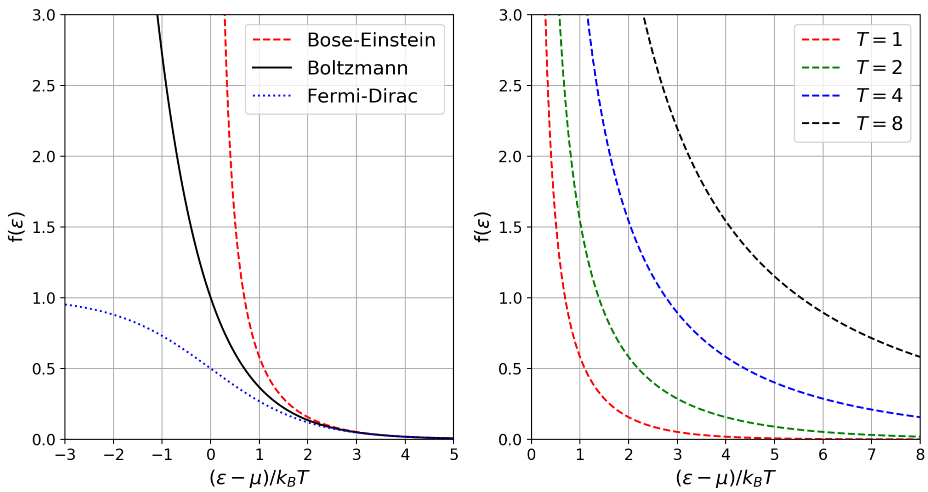

Figure 2: The different distribution functions are shown on the left, and the Bose-Einstein distributions with different temperature parameters are shown on the right. The x-axis is the dimensionless parameter $(\epsilon - \mu)/k_B T$, and the y-axis is the probability density. The figures are inspired from Carl Paterson's Statistical Physics lectures.

Bose-Einstein Distribution

Just as a classical gas obeys the Boltzmann distribution, a system of indistinguishable bosons obeys the Bose-Einstein distribution. It will suffice to simply state the mathematical form of the distribution, which is as follows:

$$p(\epsilon) = f(\epsilon)d\epsilon = \frac{d\epsilon}{\mathrm{e}^{(\epsilon-\mu)/k_B T} - 1}, \tag{3}$$

where $p(\epsilon)$ is the probability that a boson occupies a state with energy ϵ, μ is the chemical potential, k_B is the Boltzmann constant and T is the temperature of the system. It differs from the Boltzmann distribution only by the -1 in its denominator. It it then obvious that in the classical limit $(\epsilon - \mu)/k_B T \gg 1$, it approximates to the Boltzmann distribution. The deviation from the classical Boltzmann distribution becomes more severe as the parameter $(\epsilon - \mu)/k_B T$ becomes smaller. This is shown on Figure 2.

As the temperature increases, the distribution becomes wider towards higher energy states. So, it is clear that if the temperature is low enough, most of the particles will occupy the lower energy states. In the limit, *all particles will occupy the ground state.* This is the foundation of Bose-Einstein condensation.

Bose-Einstein Condensation

We have yet to give a precise definition of a Bose-Einstein condensate. All we have done so far was to *imply* that a certain number of constituents of a bosonic system accumulate on the ground state on a relatively macroscopic scale. There are various criterion that exist which can help us formulate precisely when a substance is a Bose-Einstein condensate.

Criteria

The first criterion is proposed by Einstein, and is as follows: in the thermodynamic limit that the particle number N and the volume V of the system both tend to infinity, while their ratio N/V tending towards a constant, a system is considered to be a Bose-Einstein condensate if

$$\lim_{N \to \infty} \frac{N_0}{N} > 0,$$

where N_0 is the number of particles in the ground state. [6]

Other criteria exist, such as Yang criterion and Penrose-Onsager criterion, that provide improved conditions. They will not be covered here, but the reader is encouraged to look them up.

The process to obtain a condensate first proposed by Einstein suggested an isothermal compression, where the particle density would be increased at a constant temperature. [7] However, it is also possible to satisfy the conditions stated above by cooling the substance to extremely low temperatures, which is how they are obtained in labs. A general condition for Bose-Einstein condensates to form in a dilute gas is the thermal de Broglie wavelengths of the atoms to exceed the mean atomic spacing. The temperature scales required for this is extremely low, as will be seen in the section "Bose-Einstein Condensation in Dilute Atoms".

Experiments and Observations

Superfliud Helium-4

The discovery of the lambda point of helium-4 dates back to 1911, where its odd behaviour was noticed by the Dutch physicist Heike Kamerlingh Onnes. Helium-4 is a boson, and when it is cooled below its lambda point temperature, which is around 2.17 K, it becomes a *superfluid.*

Superfluids have very interesting properties, the first of which is that they have zero viscosity. They can flow through capillary tubes that normal liquids can't get through, and can climb out of a container as seen on Figure 3 due to van der Waals attraction with the walls of the container. [8]

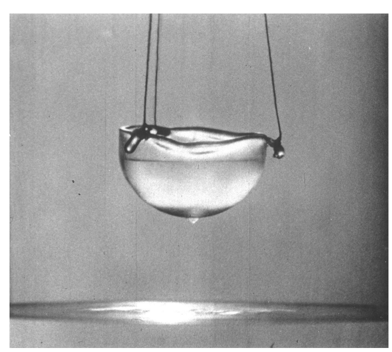

Figure 3: Helium-4 in its superfluid phase. It can be seen that it climbs along the walls of the beaker and drips down, due to its lack of viscosity. (Alfred Leitner, "Liquid helium,Superfluid", 1963)

In a demonstration film made by Alfred Leitner in 1963, helium-4 is cooled by evoporative cooling and a series of observations and experiments on its properties are made. [2] An interesting phenomenon happens at the transition temperature. The liquid boils as it is being cooled, and the boiling becomes increasingly violent as the liquid approaches the lambda point temperature. The boiling suddenly stops after it reaches the lambda point temperature, and no more bubbles can be seen. The transition is shown on Figure 4.

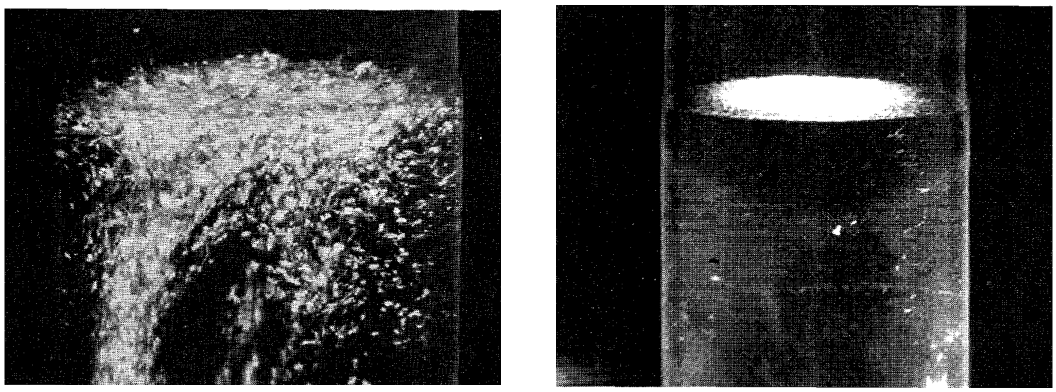

Figure 4: Helium-4 transitioning into its superfliud phase. The bubbles suddenly stop, and the liquid can no longer boil. (Alfred Leitner, "Liquid helium,Superfluid", 1963)

The reason for this is an increase in the thermal conductivity of helium-4 by a factor of about 10^6 as it transitions. Bubbles are formed as the fluid is locally vaporised, yet the thermal conductivity increases to a point which does not allow this. This means no more bubbles can form, as there needs to be a temperature gradient in various regions of the fluid. [8]

Properties such as high thermal conductivity and zero viscosity are quantum behaviour, seen on a macroscopic scale, as one would expect from a Bose-Einstein condensate. However, helium-4 is not considered a pure BEC, as the helium atoms are not weakly interacting. It can be deduced by simple experiment that helium-4 cooled below its lambda point temperature exhibits both superfluid behaviour described above, and simultaneously ordinary fluid behaviour such as finite viscosity. This is best explained by the two fluid model, where a portion of the liquid helium condenses to become a superfluid whereas a portion remains ordinary.

Superconducting Metals

Similar to superfluidity, when cooled below a certain critical temperature some conducting materials exhibit quantum behaviour at a macroscopic level. They have exactly zero electrical resistance and exhibit a phenomenon named the "Meissner Effect". Any superconductor, independent of its material composition, exhibits these behaviours after a phase transition which suggests that superconductivity is a thermodynamic state.

But the question arises, how do electrons, which are fermions, form what seems to resemble a Bose-Einstein condensate? Furthermore, how does that result in zero electrical resistance? The answers to those questions were provided by John Bardeen, Leon Cooper and J. Robert Schrieffer in 1957 with a theory of superconductivity known as the BCS theory. [9]

Electrical resistance arises due to collisions between valence electrons and the crystal lattice coupled with electrons scattering from impurities on the metal. The BCS theory proposes that electrons in the metal pair together and form what is known as a *Cooper pair.* These pairs of electrons have integer spin, so each pair obeys Bose-Einstein statistics. If the metal is then cooled below a certain temperature, the pairs accumulate in the ground state and form a single

[2] The video series can be found on www.alfredleitner.com

coherent state. The end result of this is that a single scattering event between an electron and a metal ion would require much greater energies than thermally present in the system, due to the quantum nature of the Cooper pairs. Hence, no collisions occur and the metal has exactly zero electrical resistance. [9]

One might be inclined to think that Cooper pairs are simply electrons paired up next to each other, however that is not the case. Although the pairs are related to each other, the separation distances between the electrons in a Cooper pair can be on the orders of hundreds of thousands of ions. [9] Due to quantum effects, which are too complicated for the purposes of this article, the pair effectively acts as a single boson and forms a condensate below a critical temperature.

Bose-Einstein Condensation in Dilute Atoms

C. E. Wieman and E. A. Cornell, in 1995, produced a Bose-Einstein condensate in a dilute gas of rubidium-87 atoms. This is much different from superfluid helium-4 as the number density of the rubidium gas is around 2.5×10^{12} per cubic centimeter, whereas a typical number density of a liquid is on the order 10^{23} atoms per cubic centimeter. A gas of around 2000 rubidium atoms was cooled down to 170 nK, and the condensate could be maintained for more than 15 seconds. [10]

In a paper written for the Science Magazine titled *"Observation of Bose-Einstein condensation in a Dilute Atomic Vapor"*, several reasons are stated for the specific use of alkali atoms. Firstly, light scattering can be more easily used to measure the energy and density of a cloud of gas due to easily accessible resonance lines. Furthermore, the interactions between atoms are weak and theoretically understood.

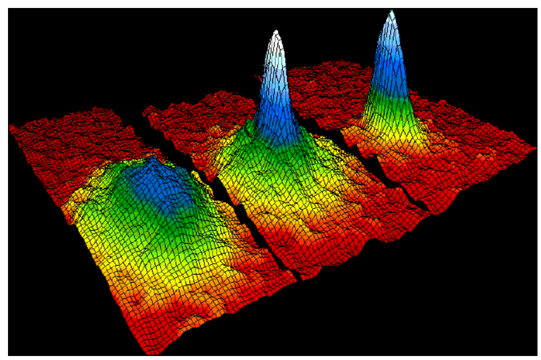

Figure 5: The velocity distribution of the rubidium gas, from left to right: Left: before the condensation happens, middle: just after the condensation, right: after evaporation a nearly pure condensate remains. Courtesy to: NIST/JILA/CU-Boulder, 1995.

The atoms were cooled with a "hybrid approach", which involved loading a laser-cooled sample of atoms into a magnetic trap where they are cooled by evaporation. [3]

Several important observations were made which confirmed the cooled gas was in fact a Bose-Einstein condensate. A sharp peak at zero velocity was seen on the velocity distribution of the gas, shown on Figure 5. As the substance was cooled, the population of the peak grew rapidly. Finally, the properties of the peak agreed with the minimum energy state of the magnetic trap, as opposed to some thermal state. With these observations, it was confirmed that the substance had become a Bose-Einstein condensate when cooled to 170 nK.

A similar experiment was conducted in 1995, in MIT by W. Ketterle. The group published a paper titled *"Bose-Einstein Condensation in a Gas of Sodium Atoms"* in the Physical Review. They managed to observe Bose-Einstein condensation in a gas that consisted up to 5×10^5 sodium atoms, confirmed by observations similar to the ones made by Cornell and Wieman. The gas was cooled, using similar techniques, down below 2 µK. [11]

[3] More details on the cooling process can be found in the paper "Observation of Bose-Einstein condensation in a Dilute Atomic Vapor" [10].

Conclusion

Quantum mechanics is a very abstract subject, describing physics on scales unimaginably small by abstract vectors in Hilbert space and functionals in its dual space. It can be confirmed by many experimental observations, yet actually *seeing* quantum behaviour is very challenging. Since systems exhibiting quantum behaviour in general are very small, measurements disturb them due to the uncertainty principle. Bose-Einstein condensates are examples of systems that exhibit quantum behaviour on macroscopic scales. In a sense, they provide a medium in which quantum phenomena can be *seen*.

Starting in 1924 with Bose's paper on statistics of photons, statistics of bosons was first formulated. 71 years after Einstein's proposal in 1925, the first pure Bose-Einstein condensate was created and studied in labs. This spans almost the entirety of the field of condensed matter physics. Bose-Einstein condensation remains as an active field of research today, and new developments are being made. It brings with itself concepts closely related to it such as superconductivity and superfluidity.

States of matter such as superfluidity and superconductivity predate the theory of Bose-Einstein condensation, yet they are deeply related to the concepts surrounding it. Especially superconductivity has very important real life applications ranging from trains to particle accelerators. The field of condensed matter physics is relatively new, and pure Bose-Einstein condensates have only been produced very recently. We can only imagine how they will impact the technology in the future, or which questions about our universe they will help us answer.

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