Classical Mechanics

Emre Özer

Notes based on Claudia de Rham's lectures at Imperial College and David Tong's notes on Classical Dynamics.

Contents

1   Rigid Body Rotation

(Summation convention sometimes used in this chapter, should be clear from context.)

1.1   Basics of rigid bodies

A body is a collection of particles. We consider a set of N particles with positions $\vec{r}_a(t)$ and masses m_a, where $a = 1, \ldots, N$. Each particle obeys the equation of motion

$$m_a \ddot{\vec{r}}_a = \sum_{b \neq a} \vec{F}_{ab} + \vec{F}_a^{\text{ext}}, \tag{1.1}$$

where $\vec{F}_{ab}$ is the force on particle a due to b and $\vec{F}_a^{\text{ext}}$ is the sum of all external forces on a. It is mathematically convenient to define $\vec{F}_{aa} \equiv 0$ and to take the sum over all b instead of over $b \neq a$.

The total mass M of the body is

$$M = \sum_a m_a. \tag{1.2}$$

The centre of mass $\vec{R}$ is defined as

$$\vec{R} = \frac{1}{M} \sum_a m_a \vec{r}_a \quad \Rightarrow \quad \dot{\vec{R}} = \frac{1}{M} \sum_a m_a \dot{\vec{r}}_a, \tag{1.3}$$

which is just a weighted average with weights of m_a/M. In the continuum limit, we have

$$m_a \to \rho(\vec{r}) \quad \Rightarrow \quad M = \int_V \rho(\vec{r}) \, d^3r, \quad \text{and} \quad \vec{R} = \frac{1}{M} \int_V \rho(\vec{r}) \vec{r} \, d^3r. \tag{1.4}$$

Similarly, the total momentum $\vec{P}$ of the body is

$$\vec{P} = \sum_a \vec{p}_a = \sum_a m_a \dot{\vec{r}}_a = M \dot{\vec{R}}. \tag{1.5}$$

It then follows that

$$\dot{\vec{P}} = \sum_a m_a \ddot{\vec{r}}_a = \sum_a \left(\sum_b \vec{F}_{ab} + \vec{F}_a^{\text{ext}} \right) = \sum_{ab} \vec{F}_{ab} + \sum_a \vec{F}_a^{\text{ext}}. \tag{1.6}$$

By Newton's third law, we have

$$\vec{F}_{ab} = -\vec{F}_{ba} \quad \Rightarrow \quad \sum_{ab} \vec{F}_{ab} \equiv 0. \tag{1.7}$$

Hence, we have

$$\dot{\vec{P}} = M \ddot{\vec{R}} = \sum_a \vec{F}_a^{\text{ext}}. \tag{1.8}$$

So, total momentum of the body is conserved if no external forces act on the body with internal forces having no effect.

The total angular momentum $\vec{L}$ is

$$\vec{L} = \sum_a \vec{\ell}_a = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a m_a \vec{r}_a \times \dot{\vec{r}}_a. \tag{1.9}$$

The rate of change of total angular momentum is then

$$
\begin{aligned}
\dot{\vec{L}} = \sum_a m_a \frac{d}{dt}\left(\vec{r}_a \times \dot{\vec{r}}_a\right) &= \sum_a m_a \left(\dot{\vec{r}}_a \times \dot{\vec{r}}_a + \vec{r}_a \times \ddot{\vec{r}}_a\right) \\
&= \sum_a m_a \vec{r}_a \times \ddot{\vec{r}}_a \\
&= \sum_a \vec{r}_a \times \left(\sum_b \vec{F}_{ab} + \vec{F}_a^{\text{ext}}\right) \\
&= \sum_{ab} \vec{r}_a \times \vec{F}_{ab} + \sum_a \vec{r}_a \times \vec{F}_a^{\text{ext}}.
\end{aligned}
\tag{1.10}
$$

Let's inspect the first term. It follows from Newton's third law that

$$
\sum_{ab} \vec{r}_a \times \vec{F}_{ab} = -\sum_{ab} \vec{r}_a \times \vec{F}_{ba}.
$$

Since the indices a and b are dummy indices, we can relabel $a \leftrightarrow b$ to obtain

$$
\sum_{ab} \vec{r}_a \times \vec{F}_{ab} = \sum_{ab} -\vec{r}_b \times \vec{F}_{ab} \quad \Rightarrow \quad \sum_{ab} \vec{r}_a \times \vec{F}_{ab} = \frac{1}{2}\sum_{ab}(\vec{r}_a - \vec{r}_b) \times \vec{F}_{ab}.
\tag{1.11}
$$

If the inter-particle forces are *central*, the force $\vec{F}_{ab}$ points in the direction of the separation vector $\vec{r}_{ab} \equiv \vec{r}_a - \vec{r}_b$. Hence, the contribution above becomes identically zero and we get

$$
\dot{\vec{L}} = \sum_a \vec{r}_a \times \vec{F}_a^{\text{ext}} \equiv \vec{\tau},
\tag{1.12}
$$

where $\vec{\tau}$ is the total torque on the body due to external forces. We again see that the inter-particle forces are *averaged out* as far as the total angular momentum is concerned.

Finally, we look at how the motion of the body decouples into centre of mass motion and relative motion about the centre of mass. We define the position relative to the centre of mass:

$$
\vec{r}^* = \vec{r} - \vec{R}.
\tag{1.13}
$$

Then, we have

$$
\sum_a m_a \vec{r}_a^* = \sum_a m_a \left(\vec{r}_a - \vec{R}\right) = M\vec{R} - M\vec{R} \equiv 0,
\tag{1.14}
$$

as expected. The total linear momentum about the centre of mass is also identically zero,

$$
\vec{P}^* = \sum_a m_a \dot{\vec{r}}_a^* = \sum_a m_a(\dot{\vec{r}}_a - \dot{\vec{R}}) = M\dot{\vec{R}} - M\dot{\vec{R}} \equiv 0 \quad \Rightarrow \quad \dot{\vec{P}}^* \equiv 0.
\tag{1.15}
$$

The total angular momentum is expressed in terms of $\vec{r}_a^*$ as follows:

$$
\begin{aligned}
\vec{L} = \sum_a m_a \vec{r}_a \times \dot{\vec{r}}_a &= \sum_a m_a \left(\vec{r}_a^* + \vec{R}\right) \times \left(\dot{\vec{r}}_a^* + \dot{\vec{R}}\right) \\
&= \sum_a m_a \left[\vec{r}_a^* \times \dot{\vec{r}}_a^* + \vec{r}_a^* \times \dot{\vec{R}} + \vec{R} \times \dot{\vec{r}}_a^* + \vec{R} \times \dot{\vec{R}}\right] \\
&= \sum_a m_a \vec{r}_a^* \times \dot{\vec{r}}_a^* + \left(\sum_a m_a \vec{r}_a^*\right) \times \dot{\vec{R}} + \vec{R} \times \left(\sum_a m_a \dot{\vec{r}}_a^*\right) + M\vec{R} \times \dot{\vec{R}} \\
&= \sum_a m_a \vec{r}_a^* \times \dot{\vec{r}}_a^* + \vec{R} \times \vec{P} \\
&\equiv \vec{L}^* + \vec{R} \times \vec{P},
\end{aligned}
\tag{1.16}
$$

where we defined the angular momentum about the central mass $\vec{L}^* = \sum_a m_a \vec{r}_a^* \times \dot{\vec{r}}_a^*$. The rate of change of $\vec{L}^*$ is given by

$$
\begin{aligned}
\dot{\vec{L}}^* = \sum_a m_a \vec{r}_a^* \times \ddot{\vec{r}}_a^* &= \sum_a m_a \left(\vec{r}_a - \vec{R} \right) \times \left(\ddot{\vec{r}}_a - \ddot{\vec{R}} \right) \\
&= \sum_a m_a \vec{r}_a \times \ddot{\vec{r}}_a - \left(\sum_a m_a \vec{r}_a \right) \times \ddot{\vec{R}} - \vec{R} \times \left(\sum_a m_a \ddot{\vec{r}}_a \right) + M \vec{R} \times \ddot{\vec{R}} \\
&= \sum_a m_a \vec{r}_a \times \ddot{\vec{r}}_a - M \vec{R} \times \ddot{\vec{R}} \\
&= \sum_a \vec{r}_a \times \vec{F}_a^{\text{ext}} = \vec{R} \times \left(\sum_a \vec{F}_a^{\text{ext}} \right) \\
&= \sum_a \left(\vec{r}_a - \vec{R} \right) \times \vec{F}_a^{\text{ext}} \\
&= \sum_a \vec{r}_a^* \times \vec{F}_a^{\text{ext}},
\end{aligned}
\tag{1.17}
$$

where we assumed central inter-particle forces.

The total kinetic energy of the body is

$$
\begin{aligned}
T = \frac{1}{2} \sum_a m_a \left(\dot{\vec{r}}_a \cdot \dot{\vec{r}}_a \right) &= \frac{1}{2} \sum_a m_a \left(\dot{\vec{r}}_a^* + \dot{\vec{R}} \right) \cdot \left(\dot{\vec{r}}_a^* + \dot{\vec{R}} \right) \\
&= \frac{1}{2} \sum_a m_a \left(\dot{\vec{r}}_a^* \cdot \dot{\vec{r}}_a^* + 2 \dot{\vec{r}}_a^* \cdot \dot{\vec{R}} + \dot{\vec{R}} \cdot \dot{\vec{R}} \right) \\
&= \frac{1}{2} \sum_a m_a \dot{\vec{r}}_a^* \cdot \dot{\vec{r}}_a^* + \frac{1}{2} M \dot{\vec{R}} \cdot \dot{\vec{R}}.
\end{aligned}
\tag{1.18}
$$

The first term is the kinetic energy about the centre of mass and the second term the kinetic energy of the centre of mass. Hence, we see that both the kinetic energy and the angular momentum decouple into a centre of mass part and a part about the centre of mass.

We conclude the section by stating that for a rigid body, $|\vec{r}_a - \vec{r}_b| \equiv 0$ for all a and b by definition.

1.2   Rotation

Consider a rigid body rotating about a point P. Define a fixed space frame $\{\tilde{\hat{e}}_a\}$ and moving body frame $\{\hat{e}_a(t)\}$ which rotates with the body. We choose the basis vectors to be orthonormal:

$$
\tilde{\hat{e}}_a \cdot \tilde{\hat{e}}_b = \delta_{ab}, \quad \text{and} \quad \hat{e}_a(t) \cdot \hat{e}_b(t) = \delta_{ab} \ \forall \, t.
\tag{1.19}
$$

Proposition. *For all t, there exists a unique, orthogonal matrix $R(t)$ such that*

$$
\hat{e}_a(t) = R_{ab}(t) \tilde{\hat{e}}_b.
\tag{1.20}
$$

Proof. Construct $R(t)$ by the rule

$$
\hat{e}_a(t) = R_{ab}(t) \tilde{\hat{e}}_b \quad \Leftrightarrow \quad R_{ab}(t) = \hat{e}_a(t) \cdot \tilde{\hat{e}}_b.
$$

Then, by orthonormality we have

$$
\delta_{ab} = \hat{e}_a(t) \cdot \hat{e}_b(t) = R_{ac}(t) \tilde{\hat{e}}_c \cdot R_{bd}(t) \tilde{\hat{e}}_d = R_{ac}(t) R_{bd}(t) \delta_{cd} = R_{ac} R_{bc} = R_{ac} R_{cb}^T = (RR^T)_{ab}.
$$

Since $RR^T = 1$, R is orthogonal. Finally, R is unique by construction. $\qquad\square$

So, the rotation of a rigid body may be completely specified by a 3×3 orthogonal matrix $R(t)$.

Example (Rotation about fixed axis). *Consider a rotation with constant angular velocity ω about $\tilde{\hat{e}}_z$ as depicted to the right. We have*

$$\hat{e}_x(t) = \tilde{\hat{e}}_x \cos \omega t + \tilde{\hat{e}}_y \sin \omega t,$$
$$\hat{e}_y(t) = \tilde{\hat{e}}_y \cos \omega t - \tilde{\hat{e}}_x \sin \omega t,$$
$$\hat{e}_z(t) = \tilde{\hat{e}}_z.$$

Writing this as a matrix equation we can recognise $R(t)$:

$$\begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix} = \begin{pmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\hat{e}}_x \\ \tilde{\hat{e}}_y \\ \tilde{\hat{e}}_z \end{pmatrix}.$$
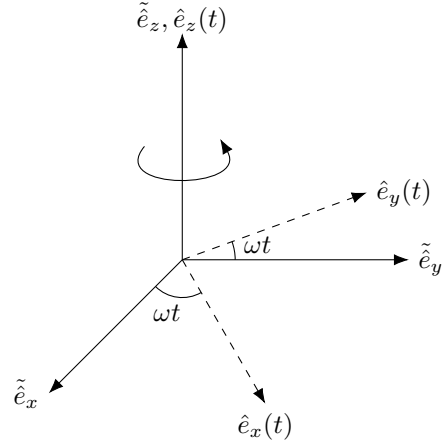
1.3 Angular velocity

Any point $\vec{r}$ in the body can be written in space or body frame:

$$\vec{r}(t) = \tilde{r}_a(t)\tilde{\hat{e}}_a \quad \text{space frame,}$$
$$= r_a\hat{e}_a(t) \quad \text{body frame.}$$

By using the rotation matrix, we can relate the components $\tilde{r}_a(t)$ and r_a as follows:

$$\vec{r} = r_a\hat{e}_a(t) = r_a R_{ab}(t)\tilde{\hat{e}}_b = \tilde{r}_b(t)\tilde{\hat{e}}_b$$
$$\Rightarrow \quad \tilde{r}_b(t) = r_a R_{ab}(t) \quad \Leftrightarrow \quad R_{ba}(t)\tilde{r}_b(t) = r_a.$$

The velocity is

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d\tilde{r}_a(t)}{dt}\tilde{\hat{e}}_a = r_a\frac{d\hat{e}_a(t)}{dt}. \tag{1.21}$$

Let's have a look at the $\dot{\hat{e}}_a(t)$ term:

$$\frac{d\hat{e}_a}{dt} = \frac{d}{dt}\left[R_{ab}(t)\tilde{\hat{e}}_b \right] = \frac{dR_{ab}}{dt}\tilde{\hat{e}}_b. \tag{1.22}$$

Substituting $\tilde{\hat{e}}_b = R_{bc}^{-1}\hat{e}_c = R_{cb}\hat{e}_c$ yields

$$\frac{d\hat{e}_a}{dt} = \dot{R}_{ab}R_{cb}\hat{e}_c \equiv \omega_{ac}\hat{e}_c, \tag{1.23}$$

where we defined $\omega_{ac} \equiv \dot{R}_{ab}R_{cb}$.

Proposition. *ω is antisymmetric.*

Proof. Since R is orthogonal, we have

$$\delta_{ac} = R_{ab}R_{cb} \quad \Rightarrow \quad 0 \equiv \dot{\delta}_{ac} = \dot{R}_{ab}R_{cb} + R_{ab}\dot{R}_{cb} = \omega_{ac} + \omega_{ca},$$

hence $\omega_{ac} = -\omega_{ca}$. $\hspace{2cm}$ $\square$

Since ω is antisymmetric, it has only 3 independent components. We can therefore define a vector with components:

$$\omega_a \equiv \frac{1}{2}\epsilon_{abc}\omega_{bc}. \tag{1.24}$$

We treat ω_a as the components of a vector in the *body frame*, so that

$$\vec{\omega} = \omega_a \hat{e}_a. \tag{1.25}$$

The reason we choose the body frame instead of the space frame will become apparent shortly.

Example (Rotation about fixed axis - again). *Referring back to the previous example, let's look at ω and $\vec{\omega}$ given $R(t)$. By definition, $\omega_{ac} = \dot{R}_{ab}R_{cb}$, hence*

$$\omega = \dot{R}R^T = \omega \begin{pmatrix} -\sin\omega t & \cos\omega t & 0 \\ -\cos\omega t & -\sin\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we can calculate components of $\vec{\omega}$ by equation (1.24), which yields

$$\vec{\omega} = \omega\hat{e}_z,$$

where note again that the components are expressed in the body frame $\{\hat{e}_a(t)\}$.

Now, let's write ω_{ac} in terms of ω_a:

$$\epsilon_{ade}\omega_a = \frac{1}{2}\epsilon_{ade}\epsilon_{abc}\omega_{bc} = \frac{1}{2}(\delta_{db}\delta_{ec} - \delta_{dc}\delta_{eb})\omega_{bc} = \frac{1}{2}(\omega_{de} - \omega_{ed}) = \omega_{de}.$$

Relabeling the indices we get:

$$\omega_{ac} = \epsilon_{bac}\omega_b = -\epsilon_{abc}\omega_b. \tag{1.26}$$

Now, we can rewrite $d\hat{e}_a/dt$ as:

$$\frac{d\hat{e}_a}{dt} = -\epsilon_{abc}\omega_b\hat{e}_c = \omega_b(\hat{e}_b \times \hat{e}_a) = \vec{\omega} \times \hat{e}_a, \tag{1.27}$$

where we used that for right-handed coordinates $\hat{e}_a \times \hat{e}_b = \epsilon_{abc}\hat{e}_c$. Obtaining this relation was the reason we defined the components of $\vec{\omega}$ to be in the body frame. The vector $\vec{\omega}$ is the *instantaneous angular velocity*.

An important corollary of relation (1.27) is that given any vector $\vec{A}$ we have

$$\dot{\vec{A}} = \dot{A}_a\hat{e}_a + A_a\dot{\hat{e}}_a = \dot{A}_a\hat{e}_a + A_a\vec{\omega} \times \hat{e}_a = \dot{A}_a\hat{e}_a + \vec{\omega} \times \vec{A}. \tag{1.28}$$

A special case is for the position vector, since $\dot{r}_a = 0$ for any point on the rotating body,

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r}. \tag{1.29}$$

1.3.1 Inertial forces

As an aside, let's not limit ourselves with a rotating rigid body but instead consider any rotating frame. Suppose our frame rotates with angular velocity $\vec{\omega}$ and we are interested in describing the motion of some particle with position $\vec{r}(t)$. By Newton's law,

$$m\ddot{\vec{r}} = \vec{F}.$$

Noting that in the rotating frame $\vec{r}(t) = r_a(t)\hat{e}_a(t)$, we have

$$\dot{\vec{r}} = \dot{r}_a\hat{e}_a + r_a\vec{\omega} \times \hat{e}_a = \vec{v}_R + \vec{\omega} \times \vec{r}, \tag{1.30}$$

where we denote $\vec{v}_R = \dot{r}_a \hat{e}_a \neq \dot{\vec{r}}$. This is not the velocity of the particle but the velocity measured in the rotating frame. Similarly,

$$
\begin{aligned}
\ddot{r} &= \ddot{r}_a \hat{e}_a + \dot{r}_a \vec{\omega} \times \hat{e}_a + \dot{r}_a \vec{\omega} \times \hat{e}_a + r_a \dot{\vec{\omega}} \times \hat{e}_a + r_a \vec{\omega} \times (\vec{\omega} \times \hat{e}_a) \\
&= \vec{a}_R + 2\vec{\omega} \times \vec{v}_R + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}),
\end{aligned}
\tag{1.31}
$$

where $\vec{a}_R$ denotes acceleration measured in the rotating frame. The equation of motion is

$$
\vec{a}_R = \frac{\vec{F}}{m} - \left(2\vec{\omega} \times \vec{v}_R + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \right).
\tag{1.32}
$$

The 3 extra terms on the right hand side are called *fictitious forces*. They arise due to the motion of our frame.

Example (Foucault pendulum). *Consider a pendulum of length ℓ and mass m, free to swing in the (x, y) plane in the coordinates defined on the right. We ignore motion along $\hat{z}$. For the Earth, we also set $\dot{\vec{\omega}} = \vec{0}$. Now, note that the centrifugal force $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ points along $\hat{z}$. The net effect of this is to reduce the effective gravitational acceleration, so let g denote this reduced value. For small angles, the equation of motion with the Coriolis force is*

$$
\ddot{r} = -\frac{g}{\ell} \vec{r} - 2\vec{\omega} \times \dot{\vec{r}},
$$

where we ignore the z component. Evaluating the cross product yields

$$
\ddot{x} - 2\Omega \dot{y} + \frac{g}{\ell} x = 0,
$$

$$
\ddot{y} + 2\Omega \dot{x} + \frac{g}{\ell} y = 0,
$$

where we defined $\omega_z = \omega \sin \lambda = \Omega$. We can rewrite this, defining $u = x + iy$ as

$$
\ddot{u} + 2i\Omega \dot{u} + \omega_0^2 u = 0
$$

where $\omega_0^2 = g/\ell$. The solution is given by

$$
u(t) = e^{-i\Omega t} \left(A e^{i\omega_1 t} + B e^{-i\omega_1 t} \right),
$$

with $\omega_1 = \sqrt{\Omega^2 + \omega_0^2}$. Notice that Ω governs the rotation of the plane of swing.
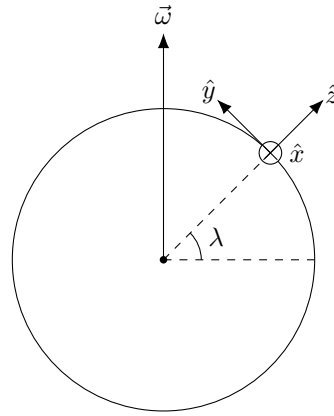
1.4 Rotation about a fixed axis

First, let's consider the simple case where the direction of $\vec{\omega}$ is fixed. Let $\vec{\omega}$ point along $\hat{e}_3$, so that $\hat{e}_3(t) = \tilde{e}_3$ for all t. The total angular momentum along $\hat{e}_3$ is

$$
L_3 = \hat{e}_3 \cdot \sum_a m_a \vec{r}_a \times (\vec{\omega} \times \vec{r}_a) = \sum_a m_a r_{a\perp}^2 \omega = I\omega,
\tag{1.33}
$$

where we defined the *moment of inertia* about $\hat{e}_3$ as $I := \sum_a m_a r_{a\perp}^2$, where $r_{a\perp}$ is shown on figure to the right. The moment of inertia arises naturally
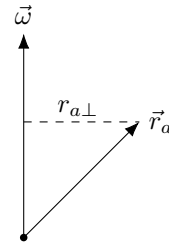
from the kinetic energy as well:

$$
\begin{aligned}
T = \frac{1}{2} \sum_a m_a \dot{\vec{r}}_a \cdot \dot{\vec{r}}_a &= \frac{1}{2} \sum_a m_a (\vec{\omega} \times \vec{r}_a) \cdot (\vec{\omega} \times \vec{r}_a) \\
&= \frac{1}{2} \sum_a m_a \big(\omega^2 r_a^2 - (\vec{\omega} \cdot \vec{r}_a)^2\big) \\
&= \frac{1}{2} \sum_a m_a \big(r_a^2 - (\vec{r}_a \cdot \hat{e}_3)^2\big)\omega^2 \\
&= \frac{1}{2} \sum_a m_a r_{a\perp}^2 \omega^2 = \frac{1}{2} I \omega^2,
\end{aligned}
\tag{1.34}
$$

where note that $r_{a\perp}^2 = r_a^2 - (\hat{e}_3 \cdot \vec{r}_a)^2$.

1.4.1   Support force

Suppose a free body rotates with some angular velocity $\vec{\omega}$. Then, it follows that the body must rotate about its centre of mass, meaning $\vec{\omega}$ should pass through the centre of mass position $\vec{R}$. This is because, in the absence of external forces the centre of mass moves with constant velocity. In the frame of reference where the centre of mass is stationary, we must have

$$
\dot{\vec{R}} = \vec{\omega} \times \vec{R} = \vec{0} \quad \Rightarrow \quad \vec{\omega} \parallel \vec{R}.
\tag{1.35}
$$

When the body doesn't rotate about its centre of mass, external forces are required to keep the rotation axis fixed. We write this condition as:

$$
\vec{F} + \vec{Q} = M\ddot{\vec{R}} = M\Big(\dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times (\vec{\omega} \times \vec{R}) + \vec{\omega} \times \vec{V}_R\Big),
\tag{1.36}
$$

where $\vec{V}_R$ denotes the centre of mass velocity as measured in the body frame. By definition, $\vec{V}_R = \vec{0}$. $\vec{F}$ is the net external force acting on the centre of mass of the body, and $\vec{Q}$ is the support force acting through the rotation axis.

Example (Compound pendulum). *Consider a compound pendulum with moment of inertia of I about $\hat{e}_z$ and mass m with centre of mass a distance R away from the pivot. The equation of motion is:*

$$
\ddot{\theta} = \frac{mgR}{I} \sin\theta,
$$

along with energy conservation:

$$
\frac{1}{2} I \dot{\theta}^2 - mgR\cos\theta = E.
$$

Note that $\vec{\omega} = \dot{\theta}\hat{e}_z$ and $\vec{R} = R\hat{e}_r$. Then, the inertial forces are:

$$
\dot{\vec{\omega}} \times \vec{R} = R\ddot{\theta}\,\hat{e}_\theta, \quad \vec{\omega} \times (\vec{\omega} \times \vec{R}) = -R\dot{\theta}^2 \hat{e}_r.
$$

Finally, in the body frame the external force is

$$
\vec{F} = mg(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta).
$$

Putting everything together and substituting for $\dot{\theta}$ and $\ddot{\theta}$, we find that the support force is

$$
\vec{Q} = mg\sin\theta\Big(1 - \frac{mR^2}{I}\Big)\hat{e}_\theta - \Big(mg\cos\theta\Big[1 + \frac{2mR^2}{I}\Big] + \frac{2mR}{I}E\Big)\hat{e}_r.
$$

This does not necessarily act radially, as the $\hat{e}_\theta$ component is non-zero for $I \neq mR^2$.
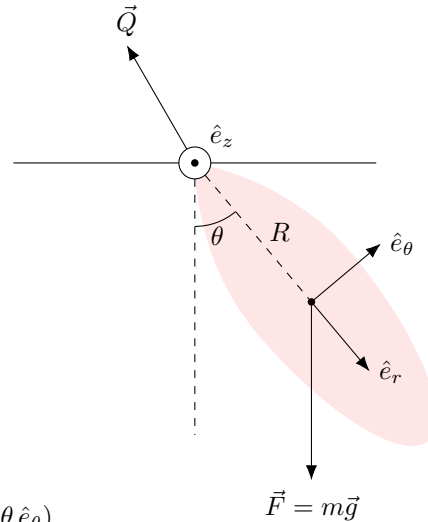
1.5 Inertia tensor

The inertia tensor generalises the moment of inertia to rotations about arbitrary axes. It again arises naturally from the angular momentum and kinetic energy. The angular momentum is

$$
\begin{aligned}
\vec{L} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i &= \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \\
&= \sum_i m_i \hat{e}_a \epsilon_{abc} r_{i,b} \epsilon_{cde} \omega_d r_{i,e} \\
&= \hat{e}_a \sum_i m_i (\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}) r_{i,b} \omega_d r_{i,e} \\
&= \hat{e}_a \sum_i m_i \big(r_i^2 \omega_a - r_{i,a} r_{i,b} \omega_b\big) \\
&= \hat{e}_a \sum_i m_i \big(r_i^2 \delta_{ab} - r_{i,a} r_{i,b}\big) \omega_b.
\end{aligned}
\tag{1.37}
$$

Hence, we see that the components of $\vec{L}$ are

$$
L_a = \sum_i m_i \big(r_i^2 \delta_{ab} - r_{i,a} r_{i,b}\big) \omega_b \equiv I_{ab}\omega_b,
\tag{1.38}
$$

where we defined the *inertia tensor* I with components

$$
I_{ab} := \sum_i m_i \big(r_i^2 \delta_{ab} - r_{i,a} r_{i,b}\big).
\tag{1.39}
$$

Now, consider the kinetic energy:

$$
\begin{aligned}
T = \frac{1}{2} \sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i &= \frac{1}{2} \sum_i m_i (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i) \\
&= \frac{1}{2} \sum_i m_i (\epsilon_{abc}\omega_b r_{i,c})(\epsilon_{ade}\omega_d r_{i,e}) \\
&= \frac{1}{2} \sum_i m_i (\delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}) \omega_b \omega_d r_{i,c} r_{i,e} \\
&= \frac{1}{2} \sum_i m_i \big(r_i^2 \delta_{bc} - r_{i,c} r_{i,b}\big) \omega_b \omega_c \\
&= \frac{1}{2} \omega_b I_{bc} \omega_c = \frac{1}{2} \vec{\omega} \cdot I \vec{\omega}.
\end{aligned}
\tag{1.40}
$$

From the definition of the inertia tensor (1.39), it follows that I is symmetric:

$$
I_{ab} = \sum_i m_i \big(r_i^2 \delta_{ab} - r_{i,a} r_{i,b}\big) = \sum_i m_i \big(r_i^2 \delta_{ba} - r_{i,b} r_{i,a}\big) = I_{ba}.
\tag{1.41}
$$

The components of the inertia tensor are measured in the body frame because we used the relation (1.29). Since the components of the position vector r_i are time-independent in the body frame, so are the components of I.

The inertia tensor generalizes to continuous bodies as follows:

$$
I = \int d^3 r\, \rho(\vec{r}) \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -yx & z^2 + x^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{pmatrix},
\tag{1.42}
$$

where we denote the position vector in the body frame as

$$\vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3.$$

Since the inertia tensor is a real, symmetric matrix, we can diagonalise it by an orthogonal transformation: $I' = OIO^T$ and $\hat{e}'_a = O\hat{e}_a$ where O is an orthogonal matrix. This would yield

$$I = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}. \tag{1.43}$$

The body axes in which I is diagonal are called the *principal axes*. Eigenvalues $I_{1,2,3}$ are called the *principal moments*. The angular momentum and kinetic energy take particularly simple forms under this choice of axes:

$$\vec{L} = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3 \quad \text{and} \quad T = \frac{1}{2}\left(I_1\omega_1^2\hat{e}_1 + I_2\omega_2^2\hat{e}_2 + I_3\omega_3^2\hat{e}_3\right). \tag{1.44}$$

If any two principal moments are equal, $I_i = I_j$, then the body is said to be *symmetric*. Then, any two perpendicular vectors lying in the plane spanned by $\hat{e}_i$ and $\hat{e}_j$ are also principal axes. To see why, set $I_2 = I_3$ and consider a rotation by angle θ around $\hat{e}_1$:

$$I' = OIO^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_2 \end{pmatrix} = I.$$

If all principal moments are equal, the body is said to be *totally symmetric*. In this case, any three mutually orthonormal vectors are principal axes.

1.5.1 Examples

Example (Rigid rod). *Consider a rigid, one dimensional rod of mass m with length ℓ along $\hat{e}_x$, as shown on the figure. The centre of mass is located at the origin. The mass density is*

$$\rho(\vec{r}) = \begin{cases} m/\ell & y = z = 0, \ |x| \leq \ell/2, \\ 0 & otherwise. \end{cases}$$
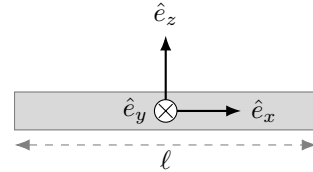
Setting $y = z = 0$ in equation (1.42) yields $I_{11} = 0$, $I_{ij} = 0$ for all $i \neq j$, and

$$I_{22} = I_{33} = \int_{-\ell/2}^{\ell/2} dx\, \frac{m}{\ell} x^2 = \frac{m\ell^2}{12}.$$

The body is symmetric, as we would expect. Since the body has no extension along $\hat{e}_y$ or $\hat{e}_z$, the principal moment along $\hat{e}_x$ is zero. As all non-diagonal terms in I are zero, $\hat{e}_{x,y,z}$ are principal axes.

Example (Line of beads). *Consider a line of N equidistant beads of equal mass along the x axis as shown on the figure. Enumerate beads by $i = 1, \ldots, N$ so that the position of the i-th bead is*

$$x_i = -\frac{\ell}{2} + \frac{\ell(i-1)}{(N-1)}, \quad y_i = z_i = 0.$$

The inertia tensor is

$$I = \sum_{i=1}^{N} \frac{M}{N} \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_i^2 & 0 \\ 0 & 0 & x_i^2 \end{pmatrix}$$
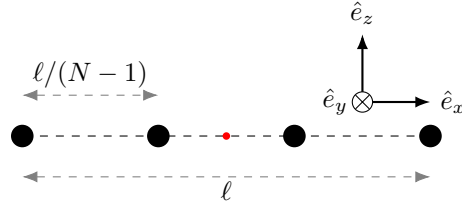
where M is the total mass. By symmetry, we again have $I_{22} = I_{33}$. Performing the sum yields

$$I_{22} = I_{33} = M\ell^2 \left(\frac{2N-1}{6(N-1)} - \frac{1}{4} \right).$$

As a sanity check, note that this is always positive for $N \geq 2$. Furthermore, in the limit $N \to \infty$, we recover the moment for rigid rod.
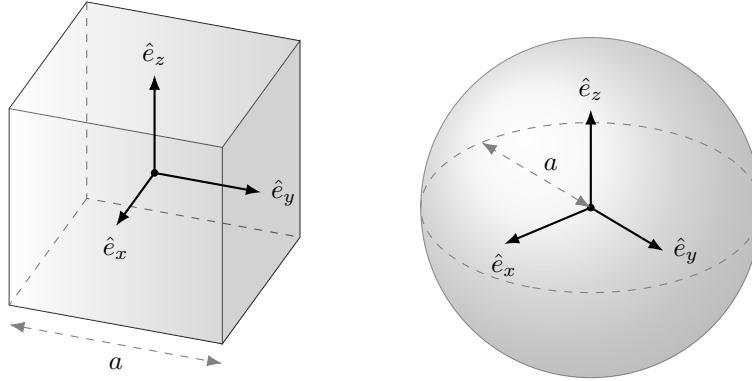
Example (Cube). *Consider cube of side length a and total mass M. The mass distribution is uniform:*

$$\rho(\vec{r}) = \begin{cases} M/a^3 & (x, y, z) \in [-a/2 \,,\, a/2]^3, \\ 0 & otherwise. \end{cases}$$

It is obvious by symmetry that $I_{11} = I_{22} = I_{33}$:

$$I_{33} = \frac{M}{a^3} \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz \, (x^2 + y^2) = \frac{2M}{a} \int_{-a/2}^{a/2} dx \, x^2 = \frac{Ma^2}{6}.$$

The cross terms evaluate to zero since the integrands are odd. The body is totally symmetric.

Example (Sphere). *Consider a sphere with radius a, total mass M. The mass distribution is uniform:*

$$\rho(\vec{r}) = \begin{cases} 3M/(\,4\pi a^3) & x^2 + y^2 + z^2 \leq a, \\ 0 & otherwise. \end{cases}$$

Clearly, by symmetry $I_{11} = I_{22} = I_{33}$:

$$I_{33} = \frac{3M}{4\pi a^3} \int_{\vec{r}^2 \leq a} d\vec{r} \, (x^2 + y^2) = \frac{2}{5} Ma^2.$$

Likewise, the cross terms evaluate to zero due to symmetry. The body is totally symmetric.

    Here is a fun thought experiment: Consider two free masses, a cube with side length ℓ and a sphere of radius $\sqrt{5/12}\,\ell$. How can we tell them apart if we can only probe their rotational and translational degrees of freedom?

1.5.2   Parallel axis theorem

So far, we have only been calculating the inertia tensor about the centre of mass. If the body is not free, it will not necessarily rotate about its centre of mass. But the inertia tensor about the centre of mass has a nice property which allows us to compute the inertia tensor about an arbitrary point more easily. We write the position vector as $\vec{r}_i = \vec{r}_i^* + \vec{R}$, where $\vec{R}$ is the centre of mass position. Then, the inertia tensor is

$$
\begin{aligned}
I_{ab} &= \sum_i m_i \left(r_i^2 \delta_{ab} - r_{i,a} r_{i,b} \right) \\
&= \sum_i m_i \left(\left[\vec{r}_i^* + \vec{R} \right]^2 \delta_{ab} - (r_{i,a}^* + R_a)(r_{i,b}^* + R_b) \right) \\
&= \sum_i m_i \left(\left[(r_i^*)^2 + 2 r_i^* R + R^2 \right] \delta_{ab} - r_{i,a}^* r_{i,b}^* - r_{i,a}^* R_b - R_a r_{i,b}^* - R_a R_b \right) \\
&= \sum_i m_i \left((r_i^*)^2 \delta_{ab} - r_{i,a}^* r_{i,b}^* \right) + 2 R \delta_{ab} \underbrace{\sum_i m_i r_i^*}_{\equiv 0} - R_b \underbrace{\sum_i m_i r_{i,a}^*}_{\equiv 0} - R_a \underbrace{\sum_i m_i r_{i,b}^*}_{\equiv 0} \\
&\quad + \sum_i m_i \left(R^2 \delta_{ab} - R_a R_b \right) \\
&= I_{ab}^* + M \left(R^2 \delta_{ab} - R_a R_b \right).
\end{aligned}
\tag{1.45}
$$

So the inertia tensor decouples into the inertia tensor about the centre of mass I^* and the inertia tensor of the centre of mass. This is known as the *parallel axis theorem.*

1.6   Euler's equations

We are interested the equation of motion for a rotating rigid body under external torque $\vec{\tau}$. As we can completely specify the rotation of the body by the rotation matrix $R(t)$, and since it is possible to obtain $R(t)$ from $\vec{\omega}(t)$, we may look for an equation relating $\vec{\omega}$ to $\vec{\tau}$. We start by stating

$$
\frac{d\vec{L}}{dt} = \frac{d}{dt}(L_a \hat{e}_a) = \dot{L}_a \hat{e}_a + L_a \vec{\omega} \times \hat{e}_a = \vec{\tau}.
\tag{1.46}
$$

Now, we work in the principal axes and write the principal moments as $I^{(1,2,3)}$ to avoid confusion with the summation convention. Then, we have $L_a = I^{(a)} \omega_a$ and so

$$
\dot{\vec{L}} = I^{(a)} \dot{\omega}_a \hat{e}_a + I^{(a)} \omega_a \omega_b \hat{e}_b \times \hat{e}_a = I^{(a)} \dot{\omega}_a \hat{e}_a + I^{(a)} \omega_a \omega_b \epsilon_{bac} \hat{e}_c.
\tag{1.47}
$$

Substituting this yields

$$
\dot{\vec{L}} = \left(I^{(c)} \dot{\omega}_c + I^{(a)} \omega_a \omega_b \epsilon_{bac} \right) \hat{e}_c = \vec{\tau}
\tag{1.48}
$$

$$
\Rightarrow \quad \dot{L}_c = I^{(c)} \dot{\omega}_c + \epsilon_{cba} \omega_a \omega_b I^{(a)} = \tau_c.
\tag{1.49}
$$

We may explicitly write the set of three equations:

$$
\dot{L}_1 = I^{(1)} \dot{\omega}_1 + \omega_2 \omega_3 \left(I^{(3)} - I^{(2)} \right) = \tau_1,
\tag{1.50}
$$

$$
\dot{L}_2 = I^{(2)} \dot{\omega}_2 + \omega_3 \omega_1 \left(I^{(1)} - I^{(3)} \right) = \tau_2,
\tag{1.51}
$$

$$
\dot{L}_3 = I^{(3)} \dot{\omega}_3 + \omega_1 \omega_2 \left(I^{(2)} - I^{(1)} \right) = \tau_3.
\tag{1.52}
$$

These are called the *Euler equations.*

1.7   Euler angles

We want an explicit parametrisation of the rotation which takes us from the space to the body frame. Any arbitrary rotation may be expressed as a composition of 3 rotations about 3 different axes. We want rotation R such that $\hat{e}_a = R_{ab}\tilde{\hat{e}}_b$. Let's split this into the three steps:

$$\{\tilde{\hat{e}}_a\} \xrightarrow{R_3(\phi)} \{\hat{e}'_a\} \xrightarrow{R_1(\theta)} \{\hat{e}''_a\} \xrightarrow{R_3(\psi)} \{\hat{e}_a\}. \tag{1.53}$$

First, we rotate by ϕ about $\tilde{\hat{e}}_3$. So, $\hat{e}'_a = R_3(\phi)_{ab}\tilde{\hat{e}}_b$ where

$$R_3(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.54}$$

Then, we rotate by θ around $\hat{e}'_1$ so that $\hat{e}''_a = R_1(\theta)_{ab}\hat{e}'_b$, where

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{1.55}$$

Finally, we rotate by ψ about $\hat{e}''_3$ so that $\hat{e}_a = R_3(\psi)_{ab}\hat{e}''_b$ where

$$R_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.56}$$

Putting everything together, we have $R(\phi, \theta, \psi) = R_3(\psi)R_1(\theta)R_3(\phi)$.

Let's now express the angular velocity in terms of the Euler angles ϕ, θ, ψ. In time dt, the angles change as

$$(\phi, \theta, \psi) \longrightarrow (\phi + d\phi, \theta + d\theta, \psi + d\psi),$$

which means the angular displacement is

$$\vec{\omega}\, dt = d\phi\, \tilde{\hat{e}}_3 + d\theta\, \hat{e}'_1 + d\psi\, \hat{e}''_3,$$

and hence the angular velocity is given by

$$\vec{\omega} = \dot{\phi}\, \tilde{\hat{e}}_3 + \dot{\theta}\, \hat{e}'_1 + \dot{\psi}\, \hat{e}''_3. \tag{1.57}$$

Now, we need to write $\tilde{\hat{e}}_3, \hat{e}'_1$ and $\hat{e}''_3$ in terms of $\hat{e}_{1,2,3}$. This is a straightforward computation:

$$\hat{e}''_3 = \hat{e}_3,$$
$$\hat{e}'_1 = \hat{e}''_1 = \cos\psi\, \hat{e}_1 - \sin\psi\, \hat{e}_2,$$
$$\tilde{\hat{e}}_3 = \hat{e}'_3 = \sin\theta\, \hat{e}''_2 + \cos\theta\, \hat{e}''_3 = \sin\theta(\sin\psi\, \hat{e}_1 + \cos\psi\, \hat{e}_2) + \cos\theta\, \hat{e}_3.$$

Hence, we have

$$\begin{aligned} \vec{\omega} &= \dot{\phi}(\sin\theta(\sin\psi\, \hat{e}_1 + \cos\psi\, \hat{e}_2) + \cos\theta\, \hat{e}_3) + \dot{\theta}(\cos\psi\, \hat{e}_1 - \sin\psi\, \hat{e}_2) + \dot{\psi}\, \hat{e}_3 \\ &= \hat{e}_1\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right) + \hat{e}_2\left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right) + \hat{e}_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right). \end{aligned} \tag{1.58}$$

2 Lagrangian Mechanics

(We use Einstein summation convention throughout this chapter.)

2.1 Generalised coordinates

Newton's laws restrict us to deal with components of vectors in inertial frames. We are interested in formulating classical mechanics in a way more general than that. A sensible first step would be to propose a way of specifying the configuration of a system.

The (spatial) configuration of any given system with (finite) N degrees of freedom is specified by a finite set of coordinates $\{q^1, q^2, \ldots, q^N\}$.[1] Such a set of coordinates is called *generalised coordinates.* For example, a set of n free particles requires $3n$ generalised coordinates to specify the position of each. A free rigid body is described by 6 coordinates: 3 for centre of mass position and 3 Euler angles.

We can go one step further and define a *configuration space* $\mathcal{C}$ as the space of all possible configurations of the system. A choice of generalised coordinates $\{q^a\}$ then becomes an explicit parametrisation of $\mathcal{C}$. The configuration of a system at any given time corresponds to a *point* in $\mathcal{C}$. As the system evolves in time, it traces out a *curve* in $\mathcal{C}$.

All we need now is a method to determine which particular curve is traversed.

2.2 Action principle

Since we are interested in finding a particular curve out of all possible curves in the configuration space, we will need to use *variational calculus.* Let's define a *functional* of the generalised coordinates such that a stationary point of the functional corresponds to the path traversed in the configuration space. This functional $\mathcal{S}[q]$, called the *action*, is defined as

$$\mathcal{S}[q] := \int_{t_i}^{t_f} L(q, \dot{q}, t)\, dt\,, \tag{2.1}$$

where $L(q, \dot{q}, t)$[2] is the *Lagrangian*, defined as the difference between the kinetic and potential energy of the system:

$$L(q, \dot{q}, t) := T(\dot{q}) - V(q, t). \tag{2.2}$$

Out of all curves in $\mathcal{C}$ with fixed endpoints

$$q(t_i) = q_i \quad \text{and} \quad q(t_f) = q_f, \tag{2.3}$$

the system follows the one which makes the action $\mathcal{S}$ stationary. This is called the *action principle.*

Note that in equation (2.1) we assumed that the curve is parametrised by time so that $q^a = q^a(t)$. Equivalently, we could have used an arbitrary parametrisation. If the Lagrangian has explicit time dependence, using another parametrisation would imply that time itself should be taken as a generalised coordinate. In relativity, since space and time are treated on an equal footing, the curve may not always be parametrised by time.

---

[1] Strictly speaking, the configuration of a system at a given time is represented by a point on an N dimensional manifold. The coordinate functions q^i provide a local chart to the manifold. To specify the complete state of the system, we need to specify velocities associated to each coordinate. This corresponds to the tangent bundle of the manifold, which itself is a $2N$ dimensional manifold. The same information is captured by the cotangent bundle. It is called the *phase space.*

[2] The set of all generalised coordinates $\{q^1, \ldots, q^N\}$ is denoted by q, and similarly $\dot{q}$ denotes the set $\{\dot{q}^1, \ldots, \dot{q}^N\}$.

2.3 Euler-Lagrange equations

To find the path which stationarises the action, we consider the variation

$$\delta\mathcal{S} = \mathcal{S}[q + \delta q] - \mathcal{S}[q], \tag{2.4}$$

where δq is a small perturbation to the trajectory. The boundary condition (2.3) implies

$$\delta q(t_i) = \delta q(t_f) \equiv 0. \tag{2.5}$$

The variation in the Lagrangian is

$$\delta L = L(q + \delta q, \dot{q} + \delta\dot{q}, t) - L(q, \dot{q}, t) = \frac{\partial L}{\partial q^a}\delta q^a + \frac{\partial L}{\partial \dot{q}^a}\delta\dot{q}^a. \tag{2.6}$$

Note that we always work up to linear order in δq. The variation in the action is then given by

$$\delta\mathcal{S} = \int_{t_i}^{t_f} \delta L\, dt = \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q^a}\delta q^a + \frac{\partial L}{\partial \dot{q}^a}\delta\dot{q}^a \right) dt\,. \tag{2.7}$$

Integrating second term by parts:

$$\int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{q}^a}\delta\dot{q}\, dt = \underbrace{\frac{\partial L}{\partial \dot{q}^a}\delta q^a \Big|_{t_i}^{t_f}}_{=0} - \int_{t_i}^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a} \right)\delta q^a\, dt = -\int_{t_i}^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a} \right)\delta q^a\, dt\,.$$

Hence, we have

$$\delta\mathcal{S} = \int_{t_i}^{t_f} \left[\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a} \right) \right]\delta q^a\, dt = 0. \tag{2.8}$$

Since this must hold for all δq, we must have

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a} \right) = 0 \quad \forall\, a. \tag{2.9}$$

This is known as the *Euler-Lagrange equation*.

As a side note, the *functional derivative* $\delta\mathcal{S}/\delta q$ is defined such that

$$\delta\mathcal{S} = \int \frac{\delta\mathcal{S}}{\delta q^a}\delta q^a\, dt\,. \tag{2.10}$$

So, for a Lagrangian which depends on $q, \dot{q}$ and t, the functional derivative is

$$\frac{\delta\mathcal{S}}{\delta q^a} = \frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a} \right). \tag{2.11}$$

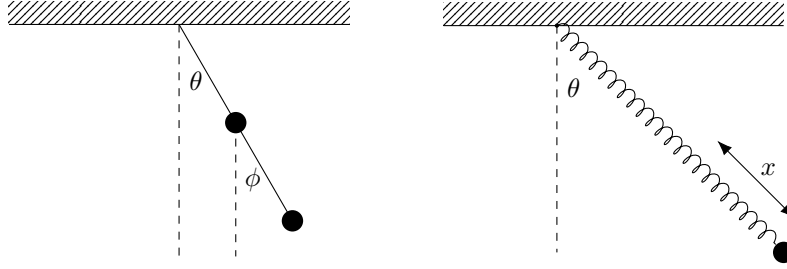We may now show that the action principle is equivalent to Newton's laws. Choosing Cartesian coordinates, we have

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

for a single particle. Then,

$$\frac{\partial L}{\partial x^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a} \right) = -m\ddot{x}_a - \frac{\partial V}{\partial x^a} = 0 \quad \Rightarrow \quad m\ddot{\vec{x}} = -\vec{\nabla}V. \tag{2.12}$$

Example (Double pendulum). *Consider two equal point masses m attached with rigid rods of length ℓ as shown from the figure. The system admits a two dimensional configuration space spanned by $(\theta, \phi) \in [0, 2\pi)^2$ (as a manifold, this is a torus). The kinetic energy and potential energies are*

$$T = m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 + m\ell^2\dot{\theta}\dot{\phi}\cos(\theta - \phi),$$
$$V = -mg\ell(2\cos\theta + \cos\phi).$$

Hence, the Lagrangian:

$$L = T - V = m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 + m\ell^2\dot{\theta}\dot{\phi}\cos(\theta - \phi) + mg\ell(2\cos\theta + \cos\phi).$$

The Euler-Lagrange equations for θ and ϕ are

$$2\ell\ddot{\theta} + \ell\ddot{\phi}\cos(\theta - \phi) + \ell\dot{\phi}^2\sin(\theta - \phi) + 2g\sin\theta = 0,$$
$$\ell\ddot{\phi} + \ell\ddot{\theta}\cos(\theta - \phi) - \ell\dot{\theta}^2\sin(\theta - \phi) + g\sin\phi = 0.$$

These form a system of coupled second-order nonlinear differential equations.

Example (Spring pendulum). *Consider point mass m and let θ denote the angle from the vertical and x the extension of the spring, ℓ the natural length of the spring, and k the spring constant. Then, we can write the kinetic and potential energy:*

$$T = \frac{1}{2}m(\ell + x)^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$$
$$V = -mg(\ell + x)\cos\theta + \frac{1}{2}kx^2.$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m(\ell + x)^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 + mg(\ell + x)\cos\theta - \frac{1}{2}kx^2.$$

Then, the Euler-Lagrange equation for θ is:

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \quad \Rightarrow \quad -g\sin\theta = 2\dot{x}\dot{\theta} + (\ell + x)\ddot{\theta}.$$

Similarly, for x we have

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \quad \Rightarrow \quad m\ddot{x} = -kx + m(\ell + x)\dot{\theta}^2 + mg\cos\theta.$$

Note. *Although Euler-Lagrange equations are useful, they are not fundamental. It is often better, especially in field theories, to vary the action directly instead of writing down Euler-Lagrange equations. Also, a lot of important results are more easily shown starting from the action. The next subsection is a good example of this.*

2.4   Coordinate transformations

The beauty of the Lagrangian formalism is *coordinate invariance.* The Euler-Lagrange equations are coordinate invariant - meaning they look the same in any coordinate system for the configuration space. This should be obvious as the action principle deals with paths in the configuration space. How we choose to parametrise the configuration space should be irrelevant. However, we will gain greater insight if we look at how functional derivatives transform.

Consider a coordinate transformation which may depend on time with non-zero Jacobian determinant:

$$\bar{q}^a = \bar{q}^a(q^1, \ldots, q^N, t) \quad \text{and} \quad q^a = q^a(\bar{q}^1, \ldots, \bar{q}^N, t). \tag{2.13}$$

The Lagrangian in the transformed coordinates is

$$\bar{L}(\bar{q}, \dot{\bar{q}}, t) = L(q(\bar{q}, t), \dot{q}(\bar{q}, \dot{\bar{q}}, t), t), \tag{2.14}$$

where

$$\dot{q}^a = \frac{dq^a}{dt} = \frac{\partial q^a}{\partial \bar{q}^b} \dot{\bar{q}}^b + \frac{\partial q^a}{\partial t}. \tag{2.15}$$

First, look at $\partial \bar{L} \big/ \partial \bar{q}^a$:

$$
\begin{aligned}
\frac{\partial \bar{L}}{\partial \bar{q}^a} &= \frac{\partial}{\partial \bar{q}^a} \left[L(q(\bar{q}, t), \dot{q}(\bar{q}, \dot{\bar{q}}, t, t)) \right] \\
&= \frac{\partial L}{\partial q^b} \frac{\partial q^b}{\partial \bar{q}^a} + \frac{\partial L}{\partial \dot{q}^b} \frac{\partial \dot{q}^b}{\partial \bar{q}^a} \\
&= \frac{\partial L}{\partial q^b} \frac{\partial q^b}{\partial \bar{q}^a} + \frac{\partial L}{\partial \dot{q}^b} \frac{\partial}{\partial \bar{q}^a} \left(\frac{\partial q^b}{\partial \bar{q}^c} \dot{\bar{q}}^c + \frac{\partial q^b}{\partial t} \right) \\
&= \frac{\partial L}{\partial q^b} \frac{\partial q^b}{\partial \bar{q}^a} + \frac{\partial L}{\partial \dot{q}^b} \left(\frac{\partial^2 q^b}{\partial \bar{q}^a \partial \bar{q}^c} \dot{\bar{q}}^c + \frac{\partial^2 q^b}{\partial \bar{q}^a \partial t} \right).
\end{aligned}
\tag{2.16}
$$

Now, look at $\partial \bar{L} \big/ \partial \dot{\bar{q}}^a$:

$$\frac{\partial \bar{L}}{\partial \dot{\bar{q}}^a} = \frac{\partial L}{\partial \dot{q}^b} \frac{\partial \dot{q}^b}{\partial \dot{\bar{q}}^a} = \frac{\partial L}{\partial \dot{q}^b} \frac{\partial}{\partial \dot{\bar{q}}^a} \left(\frac{\partial q^b}{\partial \bar{q}^c} \dot{\bar{q}}^c + \frac{\partial q^a}{\partial t} \right) = \frac{\partial L}{\partial \dot{q}^b} \frac{\partial q^b}{\partial \bar{q}^c} \delta^c_a = \frac{\partial L}{\partial \dot{q}^b} \frac{\partial q^b}{\partial \bar{q}^a}. \tag{2.17}$$

Hence, we have

$$
\begin{aligned}
\frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{\bar{q}}^a} \right) &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^b} \frac{\partial q^b}{\partial \bar{q}^a} \right) = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^b} \right) \frac{\partial q^b}{\partial \bar{q}^a} + \frac{\partial L}{\partial \dot{q}^b} \frac{d}{dt} \left(\frac{\partial q^b}{\partial \bar{q}^a} \right) \\
&= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^b} \right) \frac{\partial q^b}{\partial \bar{q}^a} + \frac{\partial L}{\partial \dot{q}^b} \left(\frac{\partial^2 q^b}{\partial \bar{q}^a \partial \bar{q}^c} \dot{\bar{q}}^c + \frac{\partial^2 q^b}{\partial \bar{q}^a \partial t} \right).
\end{aligned}
\tag{2.18}
$$

Putting (2.16) and (2.18) together, we obtain the transformation rule for the functional derivative:

$$\frac{\delta \mathcal{S}}{\delta \bar{q}^a} = \frac{\partial \bar{L}}{\partial \bar{q}^a} - \frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{\bar{q}}^a} \right) = \left[\frac{\partial L}{\partial q^b} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^b} \right) \right] \frac{\partial q^b}{\partial \bar{q}^a} = \frac{\partial q^b}{\partial \bar{q}^a} \frac{\delta \mathcal{S}}{\delta q^b}. \tag{2.19}$$

It follows from this that if the Euler-Lagrange equations hold in one set of coordinates q, it holds in all sets of coordinate related to q by some transformation of the form (2.13). But that's not it! Interestingly enough, the functional derivative transforms as a *covariant tensor!* Although none of the two terms in the functional derivative transforms as a tensor, their combination does.

Note. *We may prove the same result directly from the action by noting*

$$\mathcal{S} = \int dt\, L(q, \dot{q}, t) = \int dt\, \bar{L}(\bar{q}, \dot{\bar{q}}, t) \quad \Rightarrow \quad \delta\mathcal{S} = \int dt\, \frac{\delta\mathcal{S}}{\delta q^a}\delta q^a = \int dt\, \frac{\delta\mathcal{S}}{\delta\bar{q}^a}\delta\bar{q}^a. \tag{2.20}$$

The variations are related by:

$$\delta\bar{q}^a = \bar{q}^a(q + \delta q, t) - \bar{q}^a(q, t) = \frac{\partial\bar{q}^a}{\partial q^b}\delta q^b. \tag{2.21}$$

Combining the two:

$$\delta\mathcal{S} = \int dt\, \frac{\delta\mathcal{S}}{\delta q^a}\delta q^a = \int dt\, \frac{\delta\mathcal{S}}{\delta\bar{q}^b}\frac{\partial\bar{q}^b}{\partial q^a}\delta q^a \quad \Rightarrow \quad \frac{\delta\mathcal{S}}{\delta q^a} = \frac{\partial\bar{q}^b}{\partial q^a}\frac{\delta\mathcal{S}}{\delta\bar{q}^b}. \tag{2.22}$$

2.5   Dissipation in Lagrangian formalism

Dissipative systems are more tricky to deal with due to the fact that dissipative forces are not conservative and so not derived from potentials. Suppose we can express dissipation in form of some *dissipative forces.* Then, we can make use of the transformation property of the functional derivative to incorporate dissipative forces into the equation of motion.

   Consider a system with 3 degrees of freedom which may be expressed in Cartesian coordinates. Let $\vec{\gamma}$ be the dissipative force. The Newtonian equation of motion reads:

$$m\ddot{\vec{x}} = -\vec{\nabla}V + \vec{\gamma}, \tag{2.23}$$

where we naturally split up the forces into conservative and non-conservative components. Referring to equation (2.12), in component form:

$$m\ddot{x}_a = -\frac{\partial V}{\partial x^a} + \gamma_a \quad \Leftrightarrow \quad -\gamma_a = -m\ddot{x}_a - \frac{\partial V}{\partial x^a} = \frac{\delta\mathcal{S}}{\delta x^a}, \tag{2.24}$$

where γ_a denotes the covariant component of $\vec{\gamma}$. Now, we have a tensor equation since both terms transform covariantly. Then, the equation of motion in any coordinate system q is:

$$-\frac{\partial x^b}{\partial q^a}\gamma_b = \frac{\partial x^b}{\partial q^a}\frac{\delta\mathcal{S}}{\delta x^b} = \frac{\delta\mathcal{S}}{\delta q^a}. \tag{2.25}$$

This may be generalised for systems with more (or fewer) degrees of freedom. For constrained systems, describing the system with Cartesian coordinates may require the use of Lagrange multipliers.

I'm sure there is a better approach to deal with dissipation, this is the one I could come up with.

2.6   Holonomic constraints

Holonomic constraints are relationships between the coordinates of the form:

$$f_\alpha(x^1, \ldots, x^M, t) = 0, \quad \alpha = 1, \ldots, M - N. \tag{2.26}$$

These can be solved for N generalised coordinates $\{q^1, \ldots, q^N\}$. The system is said to have N degrees of freedom and so the configuration space is N-dimensional.

   Holonomic constraints arise naturally if we try to describe the configuration space with more coordinates than degrees of freedom. An example would be to use Cartesian coordinates (x, y) to describe a simple pendulum swinging along a plane.

We can deal with holonomic constraints without explicitly solving for the generalised coordinates. We consider instead the Lagrangian

$$L'(x, \dot{x}, \lambda, t) = L(x, \dot{x}, t) + \sum_{\alpha} \lambda_{\alpha} f_{\alpha}(x, t), \tag{2.27}$$

where we treat λ as additional coordinates. These are known as *Lagrange multipliers*. The Euler-Lagrange equation for λ_{α} reads:

$$\frac{\partial L'}{\partial \lambda_{\alpha}} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\lambda}_{\alpha}}\right) = f_{\alpha}(x, t) = 0, \tag{2.28}$$

which is the constraint itself. The Euler-Lagrange equation for the coordinates are

$$\frac{\partial L'}{\partial x^a} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a} + \sum_{\alpha} \lambda_{\alpha}\frac{\partial f_{\alpha}}{\partial x^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = 0 \quad \Leftrightarrow \quad \frac{\delta \mathcal{S}}{\delta x^a} = -\sum_{\alpha} \lambda_{\alpha}\frac{\partial f_{\alpha}}{\partial x^a}. \tag{2.29}$$

Constraint forces may be found by choosing Cartesian coordinates, then

$$m\ddot{x}^a = -\frac{\partial V}{\partial x^a} + \sum_{\alpha} \lambda_{\alpha}\frac{\partial f_{\alpha}}{\partial x^a}. \tag{2.30}$$

2.7 Lagrangian transformations

Some transformations of the Lagrangian leave the dynamics invariant, but may make the equations easier to obtain. Let's have a look at a few of them.

1. For any $\lambda \in \mathbb{R}$ and $f(t)$, the Lagrangian

$$L' = \lambda L + f(t)$$

   describes the same dynamics as the Lagrangian L. To see this, simply consider the functional derivative:

$$\frac{\delta \mathcal{S}'}{\delta q^a} = \frac{\partial L'}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}^a}\right) = \lambda\frac{\partial L}{\partial q^a} - \lambda\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = \lambda\frac{\delta \mathcal{S}}{\delta q^a}.$$

2. Adding a total derivative to the Lagrangian shifts the action by a constant which leaves the dynamics unchanged. Consider the action associated with the Lagrangian $L' = L + df/dt$:

$$\mathcal{S}' = \int_{t_i}^{t_f} L + \frac{df}{dt}\,dt = \int_{t_i}^{t_f} L\,dt + f\Big|_{t_i}^{t_f} = \mathcal{S} + \text{constant}.$$

   Note that f can be a function of time and any of the generalised coordinates and their derivatives, so that $f = f(q, \dot{q}, \ddot{q}, \ldots, t)$.

3. If the Lagrangian is *totally conserved,* meaning

$$\frac{dL}{dt} = \frac{\partial L}{\partial q^a}\dot{q}^a + \frac{\partial L}{\partial \dot{q}^a}\ddot{q}^a + \frac{\partial L}{\partial t} = 0,$$

   then *any* (nice enough) function of the Lagrangian $F(L)$ describes the same dynamics. To see why, consider varying the action $\mathcal{S}' = \int dt\, F(L)$:

$$\delta\mathcal{S}' = \int dt\, F'(L)\left(\frac{\partial L}{\partial q^a}\delta q^a + \frac{\partial L}{\partial \dot{q}^a}\delta\dot{q}^a\right)$$

$$= \int dt\,\delta q^a\left(F'(L)\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left[F'(L)\frac{\partial L}{\partial \dot{q}^a}\right]\right)$$

$$= \int dt\, F'(L)\delta q^a\left(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^a}\right]\right) - \int dt\,\delta q^a F''(L)\frac{dL}{dt}\frac{\partial L}{\partial \dot{q}^a}$$

The solution of the Euler-Lagrange equation for Lagrangian L makes the first integral vanish, meaning the transformed Lagrangian $F(L)$ also obeys its Euler-Lagrange equation provided $dL/dt = 0$. *This trick is used in relativity to consider L^2 instead of L.*

2.8   Conserved quantities

A function $F(q, \dot{q}, t)$ is called a *constant of motion* (or a *conserved quantity*) if its total time derivative vanishes,

$$\frac{dF}{dt} = \frac{\partial F}{\partial q^a}\dot{q}^a + \frac{\partial F}{\partial \dot{q}^a}\ddot{q}^a + \frac{\partial F}{\partial t} = 0, \tag{2.31}$$

along the path in configuration space taken by the system. Hence, F remains constant as the system evolves.

Suppose $\partial L/\partial q^a = 0$ for some q^a. Then, by Euler-Lagrange:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = \frac{\partial L}{\partial q^a} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \dot{q}^a} \quad \text{is conserved.} \tag{2.32}$$

The quantity $\partial L/\partial \dot{q}^a$ defines the *generalised momentum* associated with coordinate q^a:

$$p_a := \frac{\partial L}{\partial \dot{q}^a} \quad \Rightarrow \quad \dot{p}_a = \frac{\partial L}{\partial q^a}. \tag{2.33}$$

In Cartesian coordinates, p_a corresponds to *linear momentum*:

$$p_a = \frac{\partial L}{\partial \dot{x}^a} = m\dot{x}_a.$$

Now suppose the Lagrangian has no explicit time dependence. Then, the quantity

$$H = \frac{\partial L}{\partial \dot{q}^a}\dot{q}^a - L = p_a \dot{q}^a - L \tag{2.34}$$

is conserved, since

$$\frac{dH}{dt} = \frac{d}{dt}(p_a \dot{q}^a - L) = \dot{p}_a \dot{q}^a + p_a \ddot{q}^a - \frac{\partial L}{\partial \dot{q}^a}\ddot{q}^a - \frac{\partial L}{\partial q^a}\dot{q}^a - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t} = 0. \tag{2.35}$$

H is the *Hamiltonian*, usually identified with the total energy of the system.

2.9   Noether's theorem

Conserved quantities are related to continuous symmetries of the Lagrangian. First, let's define what we mean by a continuous symmetry.

Consider a transformation of the coordinates q which depends continuously on some parameter s such that

$$q^a(t) \longmapsto Q^a(s, t) = q^a(t) + s \left.\frac{\partial Q^a}{\partial s}\right|_{s=0} + \mathcal{O}(s^2). \tag{2.36}$$

The Lagrangian then transforms as

$$L(t) \longmapsto L'(t, s) = L(t) + s \left.\frac{\partial L}{\partial s}\right|_{s=0} + \mathcal{O}(s^2) \tag{2.37}$$

This transformation is said to be a *continuous symmetry* of the Lagrangian L if

$$\left.\frac{\partial L}{\partial s}\right|_{s=0} = 0 \quad \Leftrightarrow \quad L' = L + \mathcal{O}(s^2). \tag{2.38}$$

In other words, the Lagrangian remains constant to linear order as we dial up or down the parameter s. Noether's theorem states that for each such symmetry, there exists a *conserved quantity*.

The proof is as follows: Let τ be some arbitrary time interval. The variation of the action over τ due to a symmetry transformation (to first order in s) is

$$\delta \mathcal{S}_\tau = \mathcal{S}'_\tau - \mathcal{S}_\tau = \int_\tau dt \, (L' - L) = 0. \tag{2.39}$$

The same variation can also be expressed by explicitly varying L with respect to the coordinates, noting that $\delta q^a = s \, \partial Q^a / \partial s \big|_{s=0}$:

$$0 = \delta \mathcal{S}_\tau = \int_\tau dt \, \delta q^a \left(\frac{\partial L}{\partial q^a} - \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}^a} \right] \right) + \int_\tau dt \, \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^a} \delta q^a \right). \tag{2.40}$$

The first integral vanishes because we're varying on the solutions to equations of motion. Since τ was arbitrary, the integrand of the second integral must vanish, which implies

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^a} \left. \frac{\partial Q^a}{\partial s} \right|_{s=0} \right) = 0. \tag{2.41}$$

This is our conserved quantity.

2.9.1   Space translation symmetry

Suppose the Lagrangian has translation symmetry, so that

$$L(\vec{r} + s\vec{n}, \dot{\vec{r}}, t) = L(\vec{r}, \dot{\vec{r}}, t) \tag{2.42}$$

for some constant vector $\vec{n}$. Then, the associated conserved quantity is

$$\frac{\partial L}{\partial \dot{x}^a} n^a = p_a n^a = \vec{p} \cdot \vec{n}. \tag{2.43}$$

Hence, linear momentum in the $\vec{n}$ direction is conserved. This may be generalised to many particle systems.

Suppose now that we work in cylindrical coordinates (r, ϕ, z) and the Lagrangian has rotational symmetry about the z-axis, so that

$$L(r, \phi + s, z, \dot{r}, \dot{\phi}, \dot{z}, t) = L(r, \phi, z, \dot{r}, \dot{\phi}, \dot{z}, t). \tag{2.44}$$

Then, the conserved quantity is

$$\frac{\partial L}{\partial \phi} = p_\phi. \tag{2.45}$$

For a kinetic energy of the form

$$T = \frac{1}{2} m (\dot{r}^2 + \dot{z}^2) + \frac{1}{2} I \dot{\phi}^2 \quad \Rightarrow \quad p_\phi = I \dot{\phi} = L_z,$$

so the angular momentum about z is conserved.

2.9.2   Time translation and the Hamiltonian

Finally, consider time translation invariance, meaning the Lagrangian remains constant as $t \mapsto t + s$. This implies $\partial L / \partial t = 0$. We know already that this implies that the conserved quantity

is the Hamiltonian. The explicit derivation is slightly different from space translation since coordinates are parametrised by time. First, note that

$$\delta L = L' - L = s\frac{dL}{dt} + \mathcal{O}(s^2). \tag{2.46}$$

This yields

$$\delta \mathcal{S}_\tau = s \int_\tau dt \, \frac{dL}{dt}. \tag{2.47}$$

Now, note that the coordinates transform as

$$q^a(t) \longmapsto q^a(t+s) = q^a(t) + s\dot{q}^a(t), \quad \dot{q}^a(t) \longmapsto \dot{q}^a(t+s) = \dot{q}^a(t) + s\ddot{q}^a(t). \tag{2.48}$$

Then, explicitly varying the action yields

$$
\begin{aligned}
\delta \mathcal{S}_\tau &= \int_\tau dt \left(s\dot{q}^a \frac{\partial L}{\partial q^a} + s\ddot{q}^a \frac{\partial L}{\partial \dot{q}^a} \right) \\
&= \int_\tau dt \, s\dot{q}^a \left(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a} \right) \right) + s \int_\tau dt \left(\frac{\partial L}{\partial \dot{q}^a} \dot{q}^a \right) \\
&= s \int_\tau dt \left(\frac{\partial L}{\partial \dot{q}^a} \dot{q}^a \right).
\end{aligned}
\tag{2.49}
$$

Equating the two expressions for $\delta \mathcal{S}_\tau$ yields:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a} \dot{q}^a - L \right) = 0 \quad \Leftrightarrow \quad \frac{dH}{dt} = 0. \tag{2.50}$$

2.10 Kinetic matrix

Consider rewriting the set of Euler-Lagrange equation as:

$$\frac{\delta \mathcal{S}}{\delta q^a} = 0 \quad \Leftrightarrow \quad \mathcal{Z}_{ab}\ddot{q}^b + \mathcal{M}_a = 0. \tag{2.51}$$

This can be done explicitly:

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a} \right) = \frac{\partial L}{\partial q^a} - \frac{\partial^2 L}{\partial \dot{q}^b \partial \dot{q}^a}\ddot{q}^b - \frac{\partial^2 L}{\partial q^b \partial \dot{q}^a}\dot{q}^b - \frac{\partial^2 L}{\partial t \partial \dot{q}^a} \tag{2.52}$$

$$\Rightarrow \quad \mathcal{Z}_{ab} = \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b} \tag{2.53}$$

$$\Rightarrow \quad \mathcal{M}_a = \frac{\partial^2 L}{\partial q^b \partial \dot{q}^a}\dot{q}^b + \frac{\partial^2 L}{\partial t \partial \dot{q}^a} - \frac{\partial L}{\partial q^a}. \tag{2.54}$$

Provided $\det(\mathcal{Z}) \neq 0$, we can invert the equation to yield a set of coupled second order equations:

$$\ddot{q}^a = -(\mathcal{Z}^{-1})^{ab}\mathcal{M}_b. \tag{2.55}$$

Although the kinetic matrix may seem useless now, it will become useful when we talk about stability.

2.11 Stability and oscillations

We can deduce a lot about the behaviour of a system around its equilibrium points assuming the Lagrangian is of the form:

$$L(q,\dot{q}) = T(q,\dot{q}) - V(q) = \frac{1}{2}\alpha_{ab}(q)\dot{q}^a\dot{q}^b - V(q). \tag{2.56}$$

Natural systems

A system is said to be *natural* if the kinetic energy can be written as a quadratic, homogeneous function of $\dot{q}$:

$$T = \frac{1}{2}\alpha_{ab}(q)\dot{q}^a\dot{q}^b \tag{2.57}$$

for some $\alpha_{ab}(q)$ which can be a function of the coordinates q. For a single particle of mass m, the kinetic energy is given by

$$T = \frac{1}{2}mg_{ab}\dot{q}^a\dot{q}^b, \tag{2.58}$$

where $g_{ab}(q)$ is the metric tensor. We therefore see that α_{ab} contains information about the metric associated with the coordinates q, as well as the inertias of each constituent of the system. Note that α can be chosen to be *symmetric*.

Assuming a Lagrangian of the form (2.56), we see that the kinetic matrix is given by

$$\mathcal{Z}(q)_{ab} = \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b} = \frac{1}{2}\alpha_{cd}(q)\frac{\partial^2}{\partial \dot{q}^a \partial \dot{q}^b}(\dot{q}^c\dot{q}^d) = \alpha_{ab}(q). \tag{2.59}$$

From now on we will work with the kinetic matrix instead and simply write

$$T(q,\dot{q}) = \frac{1}{2}\mathcal{Z}_{ab}\dot{q}^a\dot{q}^b. \tag{2.60}$$

Equilibrium points

An *equilibrium point* q_{eq} is defined as a point in the configuration space which satisfies the Euler-Lagrange equations for all time t. In other words, the curve $q(t) = q_{\text{eq}}$ is a solution of the Euler-Lagrange equations. This implies $\dot{q} \equiv 0$. We can then find an equilibrium point by solving:

$$\left.\frac{\partial L}{\partial q^a}\right|_{q_{\text{eq}},\dot{q}\equiv 0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right)\Bigg|_{q_{\text{eq}},\dot{q}\equiv 0} = 0 \quad \forall\, a. \tag{2.61}$$

For a Lagrangian of the form (2.56), we have

$$\frac{\partial L}{\partial q^a} = \frac{1}{2}\frac{\partial \mathcal{Z}_{bc}}{\partial q^a}\dot{q}^b\dot{q}^c\bigg|_{\dot{q}\equiv 0} - \left.\frac{\partial V}{\partial q^a}\right|_{q_{\text{eq}}} = -\left.\frac{\partial V}{\partial q^a}\right|_{q_{\text{eq}}},$$

and

$$\frac{\partial L}{\partial \dot{q}^a} = \frac{1}{2}\mathcal{Z}_{ab}\dot{q}^b \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = \frac{1}{2}\frac{d\mathcal{Z}_{ab}}{dt}\dot{q}^b\bigg|_{\dot{q}\equiv 0} + \frac{1}{2}\mathcal{Z}_{ab}\ddot{q}^b\bigg|_{\dot{q}\equiv 0} = 0,$$

where $\dot{q} \equiv 0 \Rightarrow \ddot{q} \equiv 0$. Hence, an equilibrium point may be found by simply solving

$$\left.\frac{\partial V}{\partial q^a}\right|_{q_{\text{eq}}} = 0 \quad \forall\, a. \tag{2.62}$$

Small oscillations

Now, let's try to describe the system about an equilibrium point q_{eq}. We start by considering a small perturbation in the form

$$q^a(t) = q^a_{\text{eq}} + \eta^a(t) \tag{2.63}$$

for all a. Since we want to obtain equation of motion linear in η, we have to expand the Lagrangian about q_{eq} up to second order. This yields

$$L(q,\dot{q}) = \frac{1}{2}\left(\mathcal{Z}_{ab}(q_{\text{eq}}) + \left.\frac{\partial \mathcal{Z}_{ab}}{\partial q^c}\right|_{q_{\text{eq}}}\eta^c + \dots\right)\dot{\eta}^a\dot{\eta}^b - \frac{1}{2}\left.\frac{\partial^2 V}{\partial q^a \partial q^b}\right|_{q_{\text{eq}}}\eta^a\eta^b + \mathcal{O}(\eta^3), \tag{2.64}$$

where we omitted the constant term $V(q_{\text{eq}})$. Noting that $\dot{\eta} = \mathcal{O}(\eta)$, only the first constant term in the expansion for $\mathcal{Z}$ survives. Then, we have

$$\frac{\partial L}{\partial q^a} = -\frac{1}{2}\left.\frac{\partial^2 V}{\partial q^b \partial q^c}\right|_{q_{\text{eq}}} \frac{\partial}{\partial \eta^a}\left(\eta^b \eta^c\right) = -\left.\frac{\partial^2 V}{\partial q^a \partial q^b}\right|_{q_{\text{eq}}} \eta^b = -V_{ab}\eta^b,$$

where we defined $V_{ab} = \partial^2 V / \partial q^a \partial q^b$ evaluated at q_{eq}. Similarly, we have

$$\frac{\partial L}{\partial \dot{q}^a} = \frac{1}{2}\mathcal{Z}_{bc}(q^{\text{eq}})\frac{\partial}{\partial \dot{\eta}^a}\left(\dot{\eta}^b \dot{\eta}^c\right) = \mathcal{Z}_{ab}\dot{\eta}^b \tag{2.65}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = \mathcal{Z}_{ab}\ddot{\eta}^b. \tag{2.66}$$

Hence, the Euler-Lagrange equations read

$$\mathcal{Z}_{ab}\ddot{\eta}^b + V_{ab}\eta^b = 0 \quad \Rightarrow \quad \ddot{\eta}^a = -(\mathcal{Z}^{-1}V)^a{}_b \eta^b = F^a{}_b \, \eta^b, \tag{2.67}$$

where we defined the matrix $F = -\mathcal{Z}^{-1}V$. In matrix form:

$$\ddot{\vec{\eta}} = F\vec{\eta}. \tag{2.68}$$

The solution is given in terms of the eigenvectors $\vec{\mu}_i$ of F, assuming they exist with real eigenvalues $\lambda_i^2 \in \mathbb{R}$:

$$F\vec{\mu}_i = \lambda_i^2 \vec{\mu}. \tag{2.69}$$

The general solution is then given by:

$$\vec{\eta}(t) = \sum_i \vec{\mu}_i\left(A_i e^{\lambda_i t} + B_i e^{-\lambda_i t}\right), \tag{2.70}$$

where A_i and B_i are integration constants. The stability is determined by the eigenvalues λ_i^2 as follows:

1. If $\lambda_i^2 < 0$, then $\lambda_i = \pm i\omega_i$ for some real ω_i. This corresponds to simple harmonic motion with frequency ω_i and the system is stable in the $\vec{\eta} = \vec{\mu}_i$ direction.

2. If $\lambda_i^2 > 0$, the perturbation $\vec{\eta}$ grows exponentially in the $\vec{\mu}_i$ direction since $\lambda_i \in \mathbb{R}$. The system is unstable.

The eigenvectors $\vec{\mu}_i$ are called *normal modes.* The system is said to be stable around the equilibrium point if all the normal modes of oscillation are stable, meaning $\lambda_i^2 < 0$ for all i.

Example (Double pendulum)**.** *Recall the Lagrangian for the double pendulum discussed before:*

$$L \propto \ell\dot{\theta}^2 + \frac{1}{2}\ell\dot{\phi}^2 + \ell\dot{\theta}\dot{\phi}\cos(\theta - \phi) + g(2\cos\theta + \cos\phi).$$

An equilibrium point is $(\theta, \phi) = (0, 0)$*. The kinetic matrix is*

$$\mathcal{Z}(\theta, \phi) = \begin{pmatrix} 2\ell & \ell\cos(\theta - \phi) \\ \ell\cos(\theta - \phi) & \ell \end{pmatrix} \quad \Rightarrow \quad \mathcal{Z}(0, 0) = \ell\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Similarly, V *is*

$$V = g\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, we have

$$F = -\mathcal{Z}^{-1}V = -\frac{g}{\ell}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

The eigenvalues yield $\omega_\pm = 2 \pm \sqrt{2}$, with eigenvectors

$$\vec{\mu}_+ = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \quad \vec{\mu}_- = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

The ω_+ mode corresponds to the pendulums swinging in opposite directions, and the ω_- mode is pendulums swinging in phase.

2.12 Continuous systems

What happens when the system we want to describe doesn't have a discrete set of generalised coordinates, but a continuum? This is the case for any field or composite system in the continuum limit. Such systems have infinite degrees of freedom. Let's first look at a simple example.

One-dimensional string

Consider a string of length ℓ_0 with fixed endpoints at $x = (0, \ell_0)$. The state of the system is specified by the displacement from equilibrium $\phi(x, t)$ for each $x \in [0, \ell_0]$. Our set of generalised coordinates is now a continuous function of position! Let's write the Lagrangian. We use the notation

$$\partial_x \phi = \frac{\partial \phi}{\partial x}, \quad \partial_t \phi = \frac{\partial \phi}{\partial t}.$$

To find the kinetic energy, we have to be careful about how we define the mass density μ. The more thorough approach is to define it so that it is homogeneous throughout the string:

$$\mu = \frac{M}{\ell} \quad \Rightarrow \quad dm = \mu\, d\ell,$$

where M is the total mass and ℓ is the total length of the string. Note that by this definition, the mass density decreases as the string is stretched. The total length of the string is

$$\ell = \int d\ell = \int \sqrt{dx^2 + d\phi^2} = \int_0^{\ell_0} \sqrt{1 + (\partial_x \phi)^2}\, dx. \tag{2.71}$$

Then, the kinetic energy is given by

$$T = \frac{1}{2} \int (\partial_t \phi)^2\, dm = \frac{1}{2} \int \mu (\partial_t \phi)^2\, d\ell. \tag{2.72}$$

This is a horrible expression which depends both on $\partial_t \phi$ and $\partial_x \phi$ (through $d\ell$). We can simply this by assuming the displacement is small and smooth so that $\partial_x \phi \ll 1$. Then,

$$d\ell = \sqrt{1 + (\partial_x \phi)^2}\, dx = \left(1 + \frac{1}{2}(\partial_x \phi)^2 + \dots\right) dx = dx + \mathcal{O}\big((\partial_x \phi)^2\big) \approx dx, \tag{2.73}$$

$$\mu = \frac{M}{\ell} \approx \frac{M}{\ell_0} = \mu_0 = \text{constant}. \tag{2.74}$$

With this approximation, the kinetic energy is

$$T = \frac{1}{2} \int_0^{\ell_0} \mu_0 (\partial_t \phi)^2\, dx. \tag{2.75}$$

The potential energy is due to the stretching of the string and is given in terms of the *tension* k as

$$V = k(\ell - \ell_0) \tag{2.76}$$

where we assume the tension remains constant. The length difference $(\ell - \ell_0)$ is given by

$$\ell - \ell_0 = \int_0^{\ell_0} \left(1 + \frac{1}{2}(\partial_x \phi)^2 + \dots \right) dx - \ell_0 = \frac{1}{2} \int_0^{\ell_0} \left((\partial_x \phi)^2 + \mathcal{O}\big((\partial_x \phi)^4\big) \right) dx \,, \qquad (2.77)$$

where this time we have to keep terms of order $(\partial_x \phi)^2$ otherwise we get no potential. The potential is therefore

$$V = \frac{1}{2} \int_0^{\ell_0} k(\partial_x \phi)^2 \, dx \,. \qquad (2.78)$$

The Lagrangian is

$$L = T - V = \int_0^{\ell_0} \frac{1}{2} \Big(\mu(\partial_t \phi)^2 - k(\partial_x \phi)^2 \Big) dx \,, \qquad (2.79)$$

which is a *functional* of the field ϕ. The integrand is called the *Lagrangian density*, denoted $\mathscr{L}$. In terms of $\mathscr{L}$, we have

$$L = \int_0^{\ell_0} \mathscr{L} \, dx \,, \quad \text{where} \quad \mathscr{L} = \frac{1}{2} \Big(\mu(\partial_t \phi)^2 - k(\partial_x \phi)^2 \Big). \qquad (2.80)$$

The action for a general continuous system in one-dimension with Lagrangian density $\mathscr{L}$ is

$$\mathcal{S}[\phi] = \int dt \int dx \, \mathscr{L}(\phi, \partial_t \phi, \partial_x \phi), \qquad (2.81)$$

$$\Rightarrow \quad \delta\mathcal{S} = \int dt \int dx \left[\frac{\partial \mathscr{L}}{\partial \phi} \delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_t \phi)} \partial_t(\delta\phi) + \frac{\partial \mathscr{L}}{\partial(\partial_x \phi)} \partial_x(\delta\phi) \right]$$
$$= \int dt \int dx \left[\frac{\partial \mathscr{L}}{\partial \phi} - \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial(\partial_t \phi)} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial(\partial_x \phi)} \right) \right] \delta\phi = 0 \qquad (2.82)$$

$$\Rightarrow \quad \frac{\partial \mathscr{L}}{\partial \phi} - \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial(\partial_t \phi)} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial(\partial_x \phi)} \right) = 0. \qquad (2.83)$$

This is the Euler-Lagrange equation for a continuous field $\phi(x,t)$, given in terms of the Lagrangian density $\mathscr{L}(\phi, \partial_t \phi, \partial_x \phi)$. For the string, this yields

$$\mu \frac{\partial^2 \phi}{\partial t^2} - k \frac{\partial^2 \phi}{\partial x^2} = 0, \qquad (2.84)$$

which is the wave equation.

Klein-Gordon equation

We can generalise this formalism to fields ϕ of time and any number of space dimensions. Let $x^\mu = \{t, x, y, z\}$ with $\mu = 0, 1, 2, 3$. Then, $\phi = \phi(x^\mu)$ and $\partial_\mu \phi = \partial\phi/\partial x^\mu$. The Lagrangian density is in general:

$$\mathscr{L} = \mathscr{L}(\phi, \partial_\mu \phi, x^\mu).$$

The Euler-Lagrange equations take the form:

$$\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) = 0. \qquad (2.85)$$

Consider a Lagrangian density of the form

$$\mathscr{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}|\vec{\nabla}\phi|^2 - V(\phi), \tag{2.86}$$

$$\frac{\partial \mathscr{L}}{\partial \phi} = -\frac{\partial V}{\partial \phi}, \tag{2.87}$$

$$\partial_t \left(\frac{\partial \mathscr{L}}{\partial(\partial_t \phi)} \right) = \frac{\partial^2 \phi}{\partial t^2}, \tag{2.88}$$

$$\vec{\nabla} \cdot \left(\frac{\partial \mathscr{L}}{\partial(\vec{\nabla}\phi)} \right) = -\nabla^2 \phi. \tag{2.89}$$

This yields the Klein-Gordon equation:

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi - \frac{\partial V}{\partial \phi} = -\Box\phi - \frac{\partial V}{\partial \phi} = 0, \tag{2.90}$$

where we defined the d'Alembertian operator $\Box = \partial_t^2 - \nabla^2$.

3 Hamiltonian Mechanics

3.1 Legendre transform

Suppose we have a function of two variables $f = f(x, y)$. Now, introduce variable u, related to x, y by

$$u = \frac{\partial f}{\partial x}. \tag{3.1}$$

We will need to invert u for x to write some $x = x(u, v)$. This puts a constraints on our function $f(x, y)$, in the sense that $\partial f / \partial x$ needs to be a *monotonic function* of u for all y. This is equivalent to the statement that $f(x, y)$ is a convex (or concave) function of x for all y.

We want a transformation of the form:

$$f(x, y) \longmapsto g(u, y) \quad \text{where} \quad x = \frac{\partial g}{\partial u}, \tag{3.2}$$

so that u and x are treated symmetrically. In other words, u is related to x through f in the same way as x is related to u through g.

The simple substitution $g(u, y) = f(x(u, y), y)$ won't satisfy the symmetry property. So, consider:

$$g(u, y) = ux(u, y) - f(x(u, y), y) \tag{3.3}$$

$$\Rightarrow \quad \frac{\partial g}{\partial u} = x(u, y) + u\frac{\partial x(u, y)}{\partial u} - \underbrace{\frac{\partial f(x, y)}{\partial x}}_{=u} \frac{\partial x(u, y)}{\partial u} = x(u, y). \tag{3.4}$$

This satisfies our condition. Does the inverse transform exist? Consider the transform $g(u, y) \longmapsto h(x, y)$ where $x = \partial g / \partial u$:

$$h(x, y) = xu - g(u, y) = xu - (xu - f(x, y)) = f(x, y). \tag{3.5}$$

Hence, not only does the inverse transform exist, it is obtained by repeated application of the original transformation in the sense that

$$f(x, y) \xrightarrow{x \to u} g(u, y) \xrightarrow{u \to x} f(x, y). \tag{3.6}$$

This transformation is called the *Legendre transform.* We will apply it to the Lagrangian to obtain the Hamiltonian.

As a last note, we note that the Legendre transform generalises to multiple variables as follows: suppose we have a function $f(x_1, \ldots, x_n, y)$ and a set $u_1, \ldots, u_n$ with

$$u_a = u_a(x_1, \ldots, x_n, y) = \frac{\partial f}{\partial x_a} \quad \text{invertible so that} \quad x_a = x_a(u_1, \ldots, u_n, y) = x_a(u, y).$$

The Legendre transform is given by

$$g(u_1, \ldots, u_n, y) = \sum_{a=1}^{n} x_a(u, y)u_a - f(x_1(u, y), \ldots, x_n(u, y), y). \tag{3.7}$$

3.2 Hamilton's equations

The point of the Hamiltonian formalism is to reformulate the Lagrangian formalism in terms of the generalised momenta

$$p_a = \frac{\partial L}{\partial \dot{q}^a},$$

instead of the $\dot{q}^a$. So, we define the Hamiltonian $H(q, p, t)$ to be the Legendre transform of the Lagrangian $L(q, \dot{q}, t)$ with respect to $\dot{q}$:

$$H(q, p, t) = p_a \dot{q}^a - L(q, \dot{q}, t), \tag{3.8}$$

where $\dot{q}$ is eliminated by inverting the generalised momenta $p_a = \partial L/\partial \dot{q}^a$ to get $\dot{q}^a = \dot{q}^a(q, p, t)$.

There are many ways to recover the equations of motion. Again, I find it best to refer back to the action whenever possible. The action, written in terms of the Hamiltonian is now a functional of (q, p) with no explicit dependence on $\dot{q}, \dot{p}$:

$$\mathcal{S}[q, p] = \int dt \, (p_a \dot{q}^a(q, p) - H(q, p, t)). \tag{3.9}$$

Now, we vary the action by δq and δp:

$$\begin{aligned}
\delta \mathcal{S} &= \int dt \left(\dot{q}^a \delta p_a + p_a \delta \dot{q}^a - \frac{\partial H}{\partial q^a} \delta q^a - \frac{\partial H}{\partial p_a} \delta p_a \right) \\
&= \int dt \left(\delta p_a \left[\dot{q}^a - \frac{\partial H}{\partial p_a} \right] - \delta q^a \left[\dot{p}_a + \frac{\partial H}{\partial q^a} \right] \right) = 0.
\end{aligned} \tag{3.10}$$

where we assumed the boundary condition $\delta q(t_i) = \delta q(t_f) = 0$. Requiring $\delta \mathcal{S} = 0$ for all δq and δp yields *Hamilton's equations:*

$$\dot{q}^a = \frac{\partial H}{\partial p_a}, \tag{3.11}$$

$$\dot{p}_a = -\frac{\partial H}{\partial q^a}. \tag{3.12}$$

Let's look at a few simple conservation laws. If the Hamiltonian has no explicit time dependence so that $\partial H/\partial t = 0$, then H is a constant of motion:

$$\frac{dH}{dt} = \frac{\partial H}{\partial q^a} \dot{q}^a + \frac{\partial H}{\partial p_a} \dot{p}_a + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial q^a} \frac{\partial H}{\partial p_a} - \frac{\partial H}{\partial p_a} \frac{\partial H}{\partial q^a} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = 0. \tag{3.13}$$

Note that $\partial H/\partial t = -\partial L/\partial t$, so this result is consistent with the results in the previous section.

If the Lagrangian doesn't explicitly depend on some coordinate q^a, then so doesn't Hamiltonian by construction. The associated momentum p_a is conserved:

$$\dot{p}_a = -\frac{\partial H}{\partial q^a} = 0. \tag{3.14}$$

We already knew these conservation laws from the Lagrangian formalism. Later, we will see a greater class of transformations called *canonical transformations*, and associated symmetries and conserved quantities.

Example (Simple pendulum). *Recall the Lagrangian for a simple pendulum:*

$$L = \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg \cos \theta.$$

The conjugate momentum is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell\dot{\theta}^2.$$

The Hamiltonian is

$$H = p_\theta \dot{\theta} - L = \frac{p_\theta^2}{2m\ell^2} - mg \cos \theta.$$

Equations of motion are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mg \sin \theta.$$

3.3   Phase space

In the Lagrangian formalism, the time evolution is described by a set of N Euler-Lagrange equations, each of which is second order in time. In the Hamiltonian formalism, this set of N second order equations are replaced by a set of $2N$ first order equations: N for momenta and N for coordinates.

Recall that in the Lagrangian formalism, the system is described by a point in an N dimensional configuration space $\mathcal{C}$. Time evolution traces out a curve in $\mathcal{C}$. There is a subtle point here: the path in $\mathcal{C}$ depends on a set of $2N$ initial conditions $(q_0, \dot{q}_0)$. A point in $\mathcal{C}$ only gives half of the initial conditions - it doesn't specify $\dot{q}_0$. In this sense, a point in configuration space does not give a *complete* description - *the state* - of the system.

In the Hamiltonian formalism, the N dimensional configuration space is replaced by a $2N$ dimensional *phase space*. A pair (q, p) specifies a point in phase space. Since Hamilton's equations are first order in time, a point in phase space not only describes the system, but *completely* determines the time evolution. It gives a complete set of $2N$ initial conditions and completely specifies the state of the system.

As a corollary, since any point in phase space completely determines the time evolution of the system, trajectories in phase space can never intersect due to uniqueness. The evolution is said to be described by a *flow* in the phase space.

3.4   Poisson brackets

Suppose we have some function $F(q, p, t)$ defined over phase space. The total time derivative is

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q^a}\dot{q}^a + \frac{\partial F}{\partial p_a}\dot{p}_a = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q^a}\frac{\partial H}{\partial p_a} - \frac{\partial F}{\partial p_a}\frac{\partial H}{\partial q^a} = \frac{\partial F}{\partial t} + \{F, H\}, \qquad (3.15)$$

where we defined the *Poisson bracket* of two functions $f(q, p, t)$ and $g(q, p, t)$ to be

$$\{f, g\} := \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}. \qquad (3.16)$$

Let's look at some properties:

1. *Antisymmetry:* $\{f, g\} = -\{g, f\}$.

2. *Linearity:* $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$.

3. *Leibniz:* $\{fg, h\} = f\{g, h\} + g\{f, h\}$.

4. *Jacobi:* $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

A corollary of equation (3.15) is that for any $F(q, p)$, we have

$$\{F, H\} = 0 \quad \Rightarrow \quad \frac{dF}{dt} = 0 \quad \Rightarrow \quad F \text{ conserved.} \qquad (3.17)$$

For example, if H doesn't depend on some q_i, then

$$\{p_a, H\} = \frac{\partial p_a}{\partial q^b}\frac{\partial H}{\partial p_b} - \frac{\partial p_a}{\partial p_b}\frac{\partial H}{\partial q^b} = 0,$$

so p_a is a constant of motion.

Suppose F and G are constants of motion, Then, $\{F, G\}$ is also a constant of motion since

$$\{\{F, G\}, H\} = \{F, \{G, H\}\} + \{G, \{H, F\}\} = 0$$

by Jacobi. The constants of motion form a closed algebra under the Poisson bracket.

Hamilton's equations can be written in terms of Poisson brackets as

$$\dot{q}^a = \{q^a, H\}, \quad \dot{p}_a = \{p_a, H\}. \tag{3.18}$$

Finally, the Poisson brackets of the coordinates are:

$$\{q^a, q^b\} = 0, \quad \{p_a, p_b\} = 0, \quad \{q_a, p_b\} = \delta_b^a. \tag{3.19}$$

Angular momentum

Consider the angular momentum in Cartesian coordinates:

$$\vec{L} = \vec{r} \times \vec{p} \quad \Leftrightarrow \quad L_a = \epsilon_{ab}{}^c x^b p_c.$$

Let's look at the Poisson bracket structure:

$$\{L_1, L_2\} = \{x^2 p_3 - x^3 p_2, x^3 p_1 - x^1 p_3\} = \{x^2 p_3, x^3 p_1\} + \{x^3 p_2, x^1 p_3\} = -x^2 p_1 + x^1 p_2 = L_3.$$

This generalises easily to:

$$\{L_a, L_b\} = \epsilon_{ab}{}^c L_c.$$

Furthermore, we have

$$\{L^2, L_a\} = 2L^b \{L_b, L_c\} = 2L^b \epsilon_{bc}{}^a L_a = 0.$$

The similarity to quantum mechanics is not an accident.

3.5 Liouville's theorem

We will present two *almost* equivalent statements of *Liouville's theorem*: one in terms of the time evolution of a volume in phase space; the other in terms of the evolution of a density function.

3.5.1 Phase space volumes

Theorem (Liouville 1). *Consider some bounded region Ω in phase space. As time evolves, although the shape of the region may change, the total volume of it remains constant.*

Proof. Consider a $2n$ dimensional phase space. The volume element at a given point (q, p) is

$$dV = dq^1 \dots dq^n \, dp_1 \dots dp_n = d^n q \, d^n p. \tag{3.20}$$

In time δt, the coordinates transform as

$$q^a \xrightarrow{\delta t} \tilde{q}^a(q, p) = q^a + \dot{q}^a(q, p)\delta t + \mathcal{O}(\delta t^2), \quad p_a \xrightarrow{\delta t} \tilde{p}_a(q, p) = p_a + \dot{p}_a(q, p)\delta t + \mathcal{O}(\delta t^2), \tag{3.21}$$

where we note that each $\dot{q}^a$ and $\dot{p}_a$ are functions of all coordinates (q, p). The volume element transforms with the Jacobian:

$$dV \xrightarrow{\delta t} d\tilde{V} = d^n \tilde{q} \, d^n \tilde{p} = \det(J(\delta t)) \, d^n q \, d^n p, \tag{3.22}$$

where the Jacobian is given by

$$J(\delta) = \begin{pmatrix} \frac{\partial \tilde{q}_1}{\partial q_1} & \cdots & \frac{\partial \tilde{q}_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{p}_n}{\partial q_1} & \cdots & \frac{\partial \tilde{p}_n}{\partial p_n} \end{pmatrix} = \begin{pmatrix} 1 + \delta t \frac{\partial \dot{q}_1}{\partial q_1} & \cdots & \delta t \frac{\partial \dot{q}_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \delta t \frac{\partial \dot{p}_n}{\partial q_1} & \cdots & 1 + \delta t \frac{\partial \dot{p}_n}{\partial p_n} \end{pmatrix} = 1 + \delta t \begin{pmatrix} \frac{\partial \dot{q}_1}{\partial q_1} & \cdots & \frac{\partial \dot{q}_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \dot{p}_n}{\partial q_1} & \cdots & \frac{\partial \dot{p}_n}{\partial p_n} \end{pmatrix}. \tag{3.23}$$

Now, we have to take a short linear algebra aside. For any matrix M, the determinant can be expressed in terms of the eigenvalues as

$$\det(M) = \prod_i \lambda_i \quad \Rightarrow \quad \det(1 + \epsilon M) = \prod_i (1 + \epsilon \lambda_i) = 1 + \epsilon \operatorname{Tr}(M) + \mathcal{O}\big(\epsilon^2\big), \tag{3.24}$$

where we used $\sum_i \lambda_i = \operatorname{Tr}(M)$. Now, combining equations (3.23) and (3.24), we get that

$$d\tilde{V} = \left(1 + \delta t \left[\frac{\partial \dot{q}^a}{\partial q^a} + \frac{\partial \dot{p}_a}{\partial p_a}\right] + \mathcal{O}\big(\delta t^2\big)\right) d^n q \, d^n p \,. \tag{3.25}$$

By Hamilton's equations, the terms in square brackets vanish:

$$\frac{\partial \dot{q}^a}{\partial q^a} + \frac{\partial \dot{p}_a}{\partial p_a} = \frac{\partial^2 H}{\partial q^a \partial p_a} - \frac{\partial^2 H}{\partial p_a \partial q^a} = 0.$$

Hence, we get

$$d\tilde{V} = \det(J(\delta t)) \, dV = \big(1 + \mathcal{O}\big(\delta t^2\big)\big) \, dV \,. \tag{3.26}$$

Noting that $J(t = 0) = 1$, we have

$$\det(J(\delta t)) = \det(J(0)) + \mathcal{O}\big(\delta t^2\big) \quad \Rightarrow \quad \frac{d(\det(J))}{dt} = 0. \tag{3.27}$$

We're basically done at this stage. But, for completeness, we can consider the finite volume V of a region Ω:

$$V(t) = \int_\Omega dV \,,$$

which, after time δt becomes

$$V(t + \delta t) = \int_{\tilde{\Omega}} d\tilde{V} = \int_\Omega \big(1 + \mathcal{O}\big(\delta t^2\big)\big) \, dV = V(t) + \mathcal{O}\big(\delta t^2\big) \quad \Rightarrow \quad \frac{dV}{dt} = 0.$$

This completes our proof. $\square$

3.5.2   Distributions over phase space

The second, *almost* equivalent statement of the theorem is concerned with distribution functions over phase space. Suppose we have an ensemble of N identical, independent systems distributed in a $2n$ dimensional phase space by some distribution $\rho(q, p, t)$. Each system is described by a corresponding point in phase space at a given time. Also suppose the time evolution for all systems is governed by the same Hamiltonian $H(q, p, t)$.

  The normalisation condition is

$$N = \int \rho(q, p, t) \, d^n q \, d^n p \quad \forall \, t, \tag{3.28}$$

where the integral is taken over all phase space. Correspondingly, the number of systems in a volume Ω of phase space is given by

$$N_\Omega = \int_\Omega \rho(q, p, t) \, d^n q \, d^n p \,. \tag{3.29}$$

Theorem (Liouville 2). *The distribution ρ is constant along every trajectory in phase space. That is, $d\rho/dt = 0$.*

Proof. From equation (3.29), it follows that

$$\frac{dN_\Omega}{dt} = \int_\Omega \frac{\partial \rho}{\partial t} \, d^n q \, d^n p \,. \tag{3.30}$$

We can express this in terms of some *phase space flux* $\vec{J}$ so that

$$\frac{dN_\Omega}{dt} = -\oint_{\partial\Omega} \vec{J} \cdot d\vec{S} = -\int_\Omega \vec{\nabla} \cdot \vec{J} \, d^n q \, d^n p \,, \tag{3.31}$$

where we used the divergence theorem. Note that since we are dealing with a $2n$ dimensional phase space, the $\vec{\nabla}$ operator is given by

$$\vec{\nabla} = \left(\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}, \frac{\partial}{\partial p_1}, \dots \frac{\partial}{\partial p_n} \right).$$

Combining equations (3.30) and (3.31) and noting we can take an infinitesimally small region Ω yields

$$-\int_\Omega \vec{\nabla} \cdot \vec{J} \, d^n q \, d^n p = \int_\Omega \frac{\partial \rho}{\partial t} \, d^n q \, d^n p \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0.$$

This is the continuity equation for a distribution over phase space.

Since $\rho(q, p, t)$ is the number density of systems, the flux can be expressed as

$$\vec{J}(q, p, t) = \rho(q, p, t) \vec{v}(q, p, t),$$

where $\vec{v}$ is the velocity of the flow in phase space:

$$\vec{v} = \left(\dot{q}^1, \dots, \dot{q}^n, \dot{p}_1, \dots, \dot{p}_n \right) = \left(\frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q^1}, \dots, -\frac{\partial H}{\partial q^n} \right)$$

This implies

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial^2 H}{\partial q^a \partial p_a} - \frac{\partial^2 H}{\partial p_a \partial q^a} = 0.$$

Hence, we have

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\vec{v} \rho) = -\vec{v} \cdot \vec{\nabla} \rho = -\left(\dot{q}^a \frac{\partial \rho}{\partial q^a} + \dot{p}_a \frac{\partial \rho}{\partial p_a} \right) \quad \Rightarrow \quad \frac{d\rho}{dt} = 0. \tag{3.32}$$

This completes our proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Although the two theorems seem different, we can infer one from the other. Consider the number of systems in some region Ω. As we follow Ω along its flow in phase space, we expect the number of systems enclosed in the region to be conserved, that is

$$N_\Omega(t) = N_{\tilde{\Omega}}(t + \delta t) \quad \Rightarrow \quad \int_\Omega \rho(q, p, t) \, d^n q \, d^n p = \int_{\tilde{\Omega}} \rho(\tilde{q}, \tilde{p}, t + \delta t) \, d^n \tilde{q} \, d^n \tilde{p} \,. \tag{3.33}$$

By the first statement of the theorem, we have

$$d^n \tilde{q} d^n \tilde{p} = \left(1 + \mathcal{O}(\delta t^2) \right) d^n q \, d^n p \,,$$

which implies

$$\int_{\tilde{\Omega}} \rho(\tilde{q}, \tilde{p}, t + \delta t) \, d^n \tilde{q} \, d^n \tilde{p} = \int_\Omega \left(\rho(\tilde{q}, \tilde{p}, t + \delta t) + \mathcal{O}(\delta t^2) \right) d^n q \, d^n p = \int_\Omega \rho(q, p, t) \, d^n q \, d^n p \,.$$

Substituting $\tilde{q} = q + \dot{q}\delta t$ and $\tilde{p} = p + \dot{p}\delta t$ and taking Ω to be small, we conclude

$$\rho(q + \dot{q}\delta t, p + \dot{p}\delta t, t + \delta t) - \rho(q, p, t) = \mathcal{O}(\delta t^2) \quad \Rightarrow \quad \frac{d\rho}{dt} = 0.$$

The reason I've said the two statements were *almost* equivalent is that the first statement merely a statement about volume flows in phase space and doesn't introduce any distribution function. In this sense, it is more fundamental and, as it turns out, it is a direct consequence of the symplectic manifold structure of phase space.

3.5.3 Liouville equation

We can rewrite $d\rho/dt = 0$ in the form

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho}{\partial q^a} \dot{q}^a + \frac{\partial \rho}{\partial p_a} \dot{p}_a \right) = \frac{\partial \rho}{\partial p_a} \frac{\partial H}{\partial q^a} - \frac{\partial \rho}{\partial q^a} \frac{\partial H}{\partial p_a} = \{H, \rho\}. \tag{3.34}$$

This is known as the *Liouville equation.* It is a statement of Liouville's theorem in terms of the distribution function.

3.6 Canonical Transformations

Recall that the Euler-Lagrange equations were coordinate invariant, meaning they took the same form under coordinate transformations of the form $q^a \longmapsto Q^a(q)$. Now that we are dealing with the Hamiltonian formalism, we are interested in the class of transformations of the form

$$q^a \longmapsto Q^a(q, p), \quad \text{and} \quad p_a \longmapsto P_a(q, p) \tag{3.35}$$

which leave Hamilton's equations invariant. By this, we mean that we require

$$\dot{Q}^a = \frac{\partial H}{\partial P_a}, \quad \text{and} \quad \dot{P}_a = -\frac{\partial H}{\partial Q^a}. \tag{3.36}$$

Such transformations are called *canonical transformations.* We may explicitly write out this condition:

$$\dot{Q}^a = \{Q^a, H\} = \frac{\partial Q^a}{\partial q^b} \frac{\partial H}{\partial p_b} - \frac{\partial Q^a}{\partial p_b} \frac{\partial H}{\partial q^b} = \frac{\partial H}{\partial q^b} \frac{\partial q^b}{\partial P_a} + \frac{\partial H}{\partial p_b} \frac{\partial p_b}{\partial P_a} = \frac{\partial H}{\partial P_a}, \tag{3.37}$$

$$\dot{P}_a = \{P_a, H\} = \frac{\partial P_a}{\partial q^b} \frac{\partial H}{\partial p_b} - \frac{\partial P_a}{\partial p_b} \frac{\partial H}{\partial q^b} = -\frac{\partial H}{\partial q^b} \frac{\partial q^b}{\partial Q^a} - \frac{\partial H}{\partial p_b} \frac{\partial p_b}{\partial Q^a} = -\frac{\partial H}{\partial Q^a}. \tag{3.38}$$

Comparing terms gives us a set of equations relating q, p, Q, P. These put the following constraints on Q^a and P_a:

$$\frac{\partial q^b}{\partial P_a} = -\frac{\partial Q^a}{\partial p_b}, \quad \frac{\partial Q^a}{\partial q^b} = \frac{\partial p_b}{\partial P^a}, \quad \frac{\partial P_a}{\partial q^b} = -\frac{\partial p_b}{\partial Q^a}, \quad \frac{\partial P_a}{\partial p_b} = \frac{\partial q^b}{\partial Q^a}. \tag{3.39}$$

As you can see, things tend to become too tedious, too quickly with this notation. Instead, consider the $2n$-tuple $\vec{x} = (\vec{q}, \vec{p})$ and the $2n \times 2n$ matrix J:

$$J = \begin{pmatrix} 0_{n \times n} & 1_{n \times n} \\ -1_{n \times n} & 0_{n \times n} \end{pmatrix}. \tag{3.40}$$

J is an example of a *symplectic form.* Hamilton's equations now read:

$$\dot{\vec{x}} = J \frac{\partial H}{\partial \vec{x}} \quad \Leftrightarrow \quad \dot{x}_i = \sum_{j=1}^{2n} J_{ij} \frac{\partial H}{\partial x_j}. \tag{3.41}$$

Now, consider the transformation

$$\vec{x} \longmapsto \vec{y}(\vec{x}) = \left(\vec{Q}(q, p), \vec{P}(q, p) \right). \tag{3.42}$$

Then, we have

$$\dot{y}_i = \sum_j \frac{\partial y_i}{\partial x_j} \dot{x}_j = \sum_{jk} \frac{\partial y_i}{\partial x_j} J_{jk} \frac{\partial H}{\partial x_k} = \sum_{jk\ell} \frac{\partial y_i}{\partial x_j} J_{jk} \frac{\partial H}{\partial y_\ell} \frac{\partial y_\ell}{\partial x_k}. \tag{3.43}$$

Now, note that $\partial y_i / \partial x_j$ is the Jacobian associated with the transformation. Denote this by $2n \times 2n$ matrix M:

$$M = \frac{\partial \vec{y}}{\partial \vec{x}} = \begin{pmatrix} [\partial Q/\partial q]_{n \times n} & [\partial Q/\partial p]_{n \times n} \\ [\partial P/\partial q]_{n \times n} & [\partial P/\partial p]_{n \times n} \end{pmatrix}. \tag{3.44}$$

Now, we can write equation (3.43) as

$$\dot{y}_i = \sum_{jk\ell} M_{ij} J_{jk} M_{\ell k} \frac{\partial H}{\partial y_\ell} \quad \Leftrightarrow \quad \dot{\vec{y}} = \left(MJM^T \right) \frac{\partial H}{\partial \vec{y}}. \tag{3.45}$$

For the transformation to be canonical, we require

$$\dot{\vec{y}} = J \frac{\partial H}{\partial \vec{y}} \quad \Rightarrow \quad MJM^T = J. \tag{3.46}$$

So, $\vec{x} \longmapsto \vec{y}$ is a canonical transformation if and only if $MJM^T = J$. This is know as the *symplectic* condition. Matrices M which obey this are said to be *symplectic*.

Let's have a closer look at the symplectic condition. Suppose M is any matrix of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where each entry A, B, C, D is an $n \times n$ matrix. Then,

$$MJM^T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} B & D \\ -A & -C \end{pmatrix} = \begin{pmatrix} 0 & AD - BC \\ BC - AD & 0 \end{pmatrix}.$$

Then, our requirement simply reduces to $AD - BC = 1$. Expressed in terms of the Jacobian, this is

$$\frac{\partial Q^a}{\partial q^b} \frac{\partial P_b}{\partial p_c} - \frac{\partial Q^a}{\partial p_b} \frac{\partial P_b}{\partial q_c} = \delta^a_c. \tag{3.47}$$

3.6.1   Poisson brackets

How do Poisson brackets transform under canonical transformations? Consider the Poisson bracket of two arbitrary functions f and g:

$$\{f, g\}_{\vec{x}} = \frac{\partial f}{\partial x_i} J_{ij} \frac{\partial g}{\partial x_j} = \frac{\partial f}{\partial y_k} M_{ki} J_{ij} \frac{\partial g}{\partial y_\ell} M_{\ell j} = \frac{\partial f}{\partial y_k} \left(MJM^T \right)_{k\ell} \frac{\partial g}{\partial y_\ell} = \frac{\partial f}{\partial y_k} J_{k\ell} \frac{\partial g}{\partial y_\ell} = \{f, g\}_{\vec{y}}, \tag{3.48}$$

Hence, Poisson brackets are left invariant under canonical transformations. This can equivalently be taken as a defining property of a canonical transformation.

We can do one better and even claim the following: *A transformation is canonical if and only if the Poisson bracket structure between the coordinates and momenta are unchanged:*

$$\{Q^a, Q^b\} = \{P_a, P_b\} = 0, \quad \text{and} \quad \{Q^a, P_b\} = \delta^a_b. \tag{3.49}$$

Proof. First, note that the Poisson bracket structure between q and p can be written as

$$\{x_i, x_j\} = \sum_{k\ell} \frac{\partial x_i}{\partial x_k} J_{k\ell} \frac{\partial x_j}{\partial x_\ell} = J_{ij}.$$

Following the same method, we have

$$\{y_i, y_j\} = \left(MJM^T \right)_{ij} = J_{ij} \quad \Leftrightarrow \quad MJM^T = J. \tag{3.50}$$

This completes the proof. $\qquad \square$

Note that

$$MJM^T = \begin{pmatrix} \{Q_i, Q_j\}_{n \times n} & \{Q_i, P_j\}_{n \times n} \\ \{P_i, Q_j\}_{n \times n} & \{P_i, P_j\}_{n \times n} \end{pmatrix}. \tag{3.51}$$

So, canonical transformations preserve the Poisson bracket structure, and whenever the Poisson bracket structure is preserved, the transformation must be canonical.

Transformation of coordinates

We know from the Lagrangian formalism that under a coordinate transformation $q_i \longmapsto Q_i(q)$, the Euler-Lagrange equations are left invariant. Let's check what constraint this puts on P_a in the Hamiltonian formalism.

We refer to equation (3.47) and note that $\partial Q / \partial p \equiv 0$, so

$$\frac{\partial Q^a}{\partial q^b} \frac{\partial P_b}{\partial p_c} = \delta_c^a \quad \Rightarrow \quad P_a = \frac{\partial q^b}{\partial Q^a} p_b.$$

This also follows from $P_a = \partial L \big/ \partial \dot{Q}^a$, as we would expect.

3.6.2 Infinitesimal canonical transformations

Suppose we consider transformations of the form

$$q^a \longmapsto Q^a(q, p; \theta) \quad \text{and} \quad p_a \longmapsto P_a(q, p; \theta), \tag{3.52}$$

which depend continuously on some parameter θ. We set $\theta = 0$ to correspond to our original coordinates such that $Q^a(q, p, 0) = q^a$ and $P_a(q, p, 0) = p_a$. We can write such an infinitesimal transformation as

$$q^a \longmapsto Q^a = q^a + \theta \left. \frac{dQ^a}{d\theta} \right|_{\theta=0} + \mathcal{O}(\theta^2), \tag{3.53}$$

$$p_a \longmapsto P_a = p_a + \theta \left. \frac{dP_a}{d\theta} \right|_{\theta=0} + \mathcal{O}(\theta^2). \tag{3.54}$$

Requiring that this is canonical puts constraints on the functions $dQ^a/d\theta$ and $dP_a/d\theta$. From equation (3.47) we find that this constraint, to linear order in θ, is

$$\frac{\partial}{\partial q^b} \left(\frac{dQ^a}{d\theta} \right) = -\frac{\partial}{\partial p_b} \left(\frac{dP_a}{d\theta} \right), \tag{3.55}$$

which is satisfied if

$$\frac{dQ^a}{d\theta} = \frac{\partial G}{\partial p_a} \quad \text{and} \quad \frac{dP_a}{d\theta} = -\frac{\partial G}{\partial q^a} \tag{3.56}$$

for some function $G(q, p)$ which is called the *generator*. An infinitesimal canonical transformation must then look like:

$$q^a \longmapsto Q^a = q^a + \theta \frac{\partial G}{\partial p_a} + \mathcal{O}(\theta^2), \tag{3.57}$$

$$p_a \longmapsto P_a = p_a - \theta \frac{\partial G}{\partial q^a} + \mathcal{O}(\theta^2). \tag{3.58}$$

Note that so far we have been interpreting transformations in the *passive* sense: relabeling same points on phase space with different labels (choice of coordinates). A one-parameter family of canonical transformations can instead be viewed in an *active* sense: as taking us from one point in phase space (q, p) to another $(q(\theta), p(\theta))$. As we vary the parameter θ, we flow along

trajectories in phase space. The components of the tangent vector to the trajectories are given by

$$\frac{dq^a}{d\theta} = \frac{\partial G}{\partial p_a}, \quad \text{and} \quad \frac{dp_a}{d\theta} = -\frac{\partial G}{\partial q^a}. \tag{3.59}$$

This is just Hamilton's equations with the Hamiltonian replaced by generator G and time replaced by parameter θ. Then, time evolution itself can be thought of as a canonical transformation of coordinates

$$(q(t_0), p(t_0)) \longmapsto (q(t), p(t)),$$

generated by the Hamiltonian.

3.6.3 Noether's theorem

Consider an infinitesimal canonical transformation generated by G. The generator G is called a *symmetry* of the Hamiltonian if, under the transformation, the Hamiltonian is left invariant. To see when this happens, simply consider the variation δH:

$$\delta H = \frac{\partial H}{\partial q^a}\delta q^a + \frac{\partial H}{\partial p_a}\delta p_a = \theta\left(\frac{\partial H}{\partial q^a}\frac{\partial G}{\partial p_a} - \frac{\partial H}{\partial p_a}\frac{\partial G}{\partial q^a}\right) = \theta\{H, G\}. \tag{3.60}$$

G is a symmetry if $\{H, G\} = 0$. But we know already that $\dot{G} = \{G, H\}$. So if G is a symmetry, it is obviously a conserved quantity.

3.6.4 Generating functions

Generating functions provide a different approach to obtaining canonical transformations. Now, let's consider transformations which can have explicit time dependence so that

$$q^a \longmapsto Q^a = Q^a(q, p, t) \quad \text{and} \quad p_a \longmapsto P_a = P_a(q, p, t). \tag{3.61}$$

In general such transformations modify the Hamiltonian (as we will see). In this case, the transformations are said to be canonical if there exists a new Hamiltonian $K = K(Q, P, t)$ such that

$$\dot{Q}^a = \frac{\partial K}{\partial P_a} \quad \text{and} \quad \dot{P}_a = -\frac{\partial K}{\partial Q^a}. \tag{3.62}$$

Let's now look at the action in both coordinates:

$$\mathcal{S}[q, p] = \int p_a \dot{q}^a - H(q, p, t)\, dt \quad \text{and} \quad \mathcal{S}[Q, P] = \int P_a \dot{Q}^a - K(Q, P, t)\, dt. \tag{3.63}$$

We require that $\delta\mathcal{S}[q, p] = 0 \Leftrightarrow \delta\mathcal{S}[Q, P] = 0$. This implies

$$\lambda(p_a \dot{q}^a - H(q, p, t)) = P_a \dot{Q}^a - K(Q, P, t) + \dot{F}. \tag{3.64}$$

for some constant λ and function F. If $\lambda = 1$, the transformation is said to be canonical; otherwise it is *extended canonical*. Finally, if $Q = Q(q, p)$ and $P = P(q, p)$ without explicit time dependence, the transformation is *restricted canonical* - which is what we considered so far. Now, we only impose $\lambda = 1$.

Note that F should be a function of q, Q and t. We can infer this by looking at dF:

$$dF = \dot{F}\, dt = p_a\, dq^a - P_a\, dQ^a + (K - H)\, dt. \tag{3.65}$$

We can perform Legendre transforms to obtain functions of other phase space coordinates. If the Legendre transform is a function of one of (q, p) and one of (Q, P), it is called a *generating function*. There are four types:

1. Let $F = F_1(q, Q, t)$. This relates q to Q. Then,

$$\dot{q}^a p_a - H = \dot{Q}^a P_a - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q^a} \dot{q}^a + \frac{\partial F_1}{\partial Q^a} \dot{Q}^a. \tag{3.66}$$

Matching the terms yields

$$p_a = \frac{\partial F_1}{\partial q^a}, \quad P_a = -\frac{\partial F_1}{\partial Q^a}, \quad K = H + \frac{\partial F_1}{\partial t}. \tag{3.67}$$

This gives us the rest of the transformation $p_a = p_a(q, Q, t)$, $P_a = P_a(q, Q, t)$.

2. Now, we simply take Legendre transforms. Let $F = F_2(q, P, t) - Q^a P_a$, where F_2 relates q to P. Then,

$$\dot{q}^a p_a - H = -Q^a \dot{P}_a - K + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q^a} \dot{q}^a + \frac{\partial F_2}{\partial P_a} \dot{P}_a, \tag{3.68}$$

which yields

$$p_a = \frac{\partial F_2}{\partial q^a}, \quad Q^a = \frac{\partial F_2}{\partial P_a}, \quad K = H + \frac{\partial F_2}{\partial t}. \tag{3.69}$$

3. $F = q^a p_a + F_3(p, Q, t)$:

$$q^a = -\frac{\partial F_3}{\partial p_a}, \quad P_a = -\frac{\partial F_3}{\partial Q^a}, \quad K = H + \frac{\partial F_3}{\partial t}. \tag{3.70}$$

4. $F = q^a p_a - Q^a P_a + F_4(p, P, t)$:

$$q^a = -\frac{\partial F_4}{\partial p_a}, \quad Q^a = \frac{\partial F_4}{\partial P_a}, \quad K = H + \frac{\partial F_4}{\partial t}. \tag{3.71}$$

3.7   Integrability

Symmetries lead to conserved quantities. Conserved quantities impose additional constraints on the system. To see this, let $F(q, p)$ be a conserved quantity. Then, the constraint is of the form

$$\dot{F} = \{F, H\} = 0.$$

Since the Hamiltonian is a function of (q, p), each such symmetry gives a constraint between the coordinates and momenta. The phase space flow is constrained to be on a surface of constant F.

    If there are enough symmetries so that a solution to Hamilton's equations can be obtained by just integration, the system is said to be *integrable*.

    For a $2n$ dimensional phase space, integrability requires n independent conserved quantities F_i with the additional conditions

$$\{F_i, F_j\} = 0 \quad \forall i, j.$$

Such functions F_i and F_j are said to be *in involution*. As it turns out, the topology of the surface we are constrained on in the phase space is that of an n-torus.

3.8   Action-Angle Variables

If a system is integrable, we expect n conserved quantities. One way to naturally obtain n conserved quantities is to have n ignorable coordinates. Supposing the Hamiltonian has no explicit q dependence, the n generalised momenta are conserved:

$$\dot{p}_a = -\frac{\partial H}{\partial q^a} \equiv 0.$$

We are interested in a transformation $(q, p) \longmapsto (\theta, I)$ which satisfy:

1. New momenta are functions of the conserved quantities: $I_a = I_a(F)$.

2. New coordinates are all *ignorable* - Hamiltonian has no explicit θ dependence.

3. Transformation is restricted canonical:

$$H(q, p) \longmapsto H(\theta, I) = H(I),$$

$$\dot{q}^a = \frac{\partial H}{\partial p_a} \longmapsto \dot{\theta}^a = \frac{\partial H}{\partial I_a} = \omega^a(I),$$

$$\dot{p}_a = -\frac{\partial H}{\partial q^a} \longmapsto \dot{I}_a = -\frac{\partial H}{\partial \theta^a} \equiv 0,$$

where $\omega^a(I)$ are functions of I.

The variables (θ, I) are called the *angle-action variables*. Since I_a are constant, all angle variables evolve at *uniform* rates $\omega^a(I)$. The equations of motion are integrated to give:

$$\theta^a = \omega^a(I)t + \beta^a, \quad I_a = \text{constant}, \tag{3.72}$$

for integration constants β^a.